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Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function

Chao-Ping Chen [a,*], Richard B. Paris [b]

[a] *School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454000, Henan Province, China*
[b] *School of Engineering, Computing and Applied Mathematics, University of Abertay, Dundee DD1 1HG, UK*

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ABSTRACT

In this paper, we present some completely monotonic functions and asymptotic expansions related to the gamma function. Based on the obtained expansions, we provide new bounds for $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ and $\Gamma(x+\frac{1}{2})$.

1. Introduction

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geqslant 0 \quad (x \in I;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.1}$$

Dubourdieu [13, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (1.1). See also [16] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [48, p. 161]). The main properties of completely monotonic functions are given in [[48], Chapter IV]. We also refer to [4], where an extensive list of references on completely monotonic functions can be found.

Euler's gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \quad x > 0,$$

is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

* Corresponding author.
E-mail addresses: chenchaoping@sohu.com (C.-P. Chen), R.Paris@abertay.ac.uk (R.B. Paris).

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is known as the psi (or digamma) function. The derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\{\psi(x)\}, \quad n \in \mathbb{N},$$

are called the polygamma functions.

In this paper, we present some completely monotonic functions and asymptotic expansions related to the gamma function. Based on the obtained expansions, we provide new bounds for $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ and $\Gamma(x+\frac{1}{2})$.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Lemmas

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

The rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called Bernoulli and Euler numbers, respectively.

It follows from Problem 154 in Part I, Chapter 4, of [39] that

$$\sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!}t^{2j} < \frac{t}{e^t-1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!}t^{2j}, \tag{2.1}$$

for $t > 0$ and $m \in \mathbb{N}_0$. The inequality (2.1) can be also found in [17,40].

Lemma 1 presents an analogous result to (2.1).

Lemma 1. *For $x > 0$ and $m \in \mathbb{N}$,*

$$\sum_{j=2}^{2m+1} \frac{(1-2^{2j})B_{2j}}{j}\frac{x^{2j-1}}{(2j-1)!} < \frac{2}{e^x+1} - 1 + \frac{x}{2} < \sum_{j=2}^{2m} \frac{(1-2^{2j})B_{2j}}{j}\frac{x^{2j-1}}{(2j-1)!}, \tag{2.2}$$

where B_n $(n \in \mathbb{N}_0)$ are the Bernoulli numbers.

Proof. The noted Boole's summation formula [45, p. 17]) states for $k \in \mathbb{N}$ that

$$f(1) = \frac{1}{2}\sum_{j=0}^{k-1}\frac{E_j(1)}{j!}\left(f^{(j)}(1) + f^{(j)}(0)\right) + \frac{1}{2(k-1)!}\int_0^1 f^{(k)}(t)E_{k-1}(t)\mathrm{d}t,$$

which can be written for $m \in \mathbb{N}$ as

$$f(1) - f(0) = \sum_{j=1}^{m}\frac{E_{2j-1}(1)}{(2j-1)!}\left(f^{(2j-1)}(1) + f^{(2j-1)}(0)\right) + \frac{1}{(2m-1)!}\int_0^1 f^{(2m)}(t)E_{2m-1}(t)\mathrm{d}t. \tag{2.3}$$

Applying formula (2.3) to $f(t) = e^{xt}$, we obtain

$$-\frac{2}{e^x+1} + 1 - \frac{x}{2} = \sum_{j=2}^{m}\frac{E_{2j-1}(1)}{(2j-1)!}x^{2j-1} + \frac{x}{e^x+1}\frac{x^{2m-1}}{(2m-1)!}\int_0^1 e^{xt}E_{2m-1}(t)\mathrm{d}t. \tag{2.4}$$

It is well known (see [1, p. 804]) that

$$E_{2m+1}(1-t) = -E_{2m+1}(t) \quad \text{and} \quad E_{2m+1}\left(\frac{1}{2}\right) = 0.$$

Noting that

$$E_{4m-1}(t) > 0, \quad E_{4m+1}(t) < 0 \quad \text{for} \quad 0 < t < 1/2, \quad m = 1, 2, \ldots,$$

we deduce for $x > 0$ that

$$\int_0^1 e^{xt}E_{4m-1}(t)\mathrm{d}t = \int_0^{1/2}(e^{xt} - e^{x(1-t)})E_{4m-1}(t)\mathrm{d}t < 0,$$

and

$$\int_0^1 e^{xt}E_{4m+1}(t)\mathrm{d}t = \int_0^{1/2}(e^{xt} - e^{x(1-t)})E_{4m+1}(t)\mathrm{d}t > 0.$$