Applied Mathematics and Computation 250 (2015) 605–614

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Simple uniform exponential stability conditions for a system of linear delay differential equations

Leonid Berezansky [a], Josef Diblík [b,*,1], Zdeněk Svoboda [b,1], Zdeněk Šmarda [b,2]

[a] Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
[b] Brno University of Technology, Brno, Czech Republic

ARTICLE INFO

Keywords:
Uniform exponential stability
Linear delay differential system
Bohl–Perron theorem

ABSTRACT

Uniform exponential stability of linear systems with time varying coefficients

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)), \quad i = 1,\dots,m$$

is studied, where $t \geqslant 0$, m and r_{ij}, $i,j = 1,\dots,m$ are natural numbers, $a_{ij}^k : [0,\infty) \to \mathbb{R}$ and $h_{ij}^k : [0,\infty) \to \mathbb{R}$ are measurable functions. New explicit result is derived with the proof based on Bohl–Perron theorem. The resulting criterion has advantages over some previous ones in that, e.g., it involves no M-matrix to establish stability. Several useful and easily verifiable corollaries are deduced and examples are provided to demonstrate the advantage of the stability result over known results.

1. Introduction

In the paper uniform explicit exponential stability is investigated for the linear delay differential system with time varying coefficients

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)), \quad i = 1,\dots,m \tag{1}$$

where $t \geqslant 0$, m and r_{ij}, $i,j = 1,\dots,m$ are natural numbers, coefficients $a_{ij}^k : [0,\infty) \to \mathbb{R}$ and delays $h_{ij}^k : [0,\infty) \to \mathbb{R}$ are measurable functions (additional assumptions will be formulated later).

For the scalar case ($m = 1$), the system (1) reduces to a linear differential equation with several delays

$$\dot{x}(t) = -\sum_{k=1}^{r} a_k(t)x(h_k(t)). \tag{2}$$

* Corresponding author.
E-mail addresses: brznsky@cs.bgu.ac.il (L. Berezansky), diblik@feec.vutbr.cz, diblik.j@fce.vutbr.cz (J. Diblík), svobodaz@feec.vutbr.cz (Z. Svoboda), smarda@feec.vutbr.cz (Z. Šmarda).
[1] This author was supported by the Grant P201/11/0768 of the Czech Grant Agency (Prague).
[2] This author was supported by the Grant FEKT-S-14-2200 of Faculty of Electrical Engineering and Communication, Brno University of Technology.

http://dx.doi.org/10.1016/j.amc.2014.10.117

Eq. (2) is studied in detail, e.g., in [1–5], and a review on stability results can be found in [6]. For system (1), there are not so many results.

In the following short overview of known results we use the notion of an M-matrix. For the reader's convenience, we recall that a square matrix is called a non-singular M-matrix if all its off-diagonal elements are non-positive and its principal minors are positive. (In [7], equivalent definitions can be found.)

Asymptotic stability conditions for the autonomous case of system (1) (when $a_{ij}^k(t) \equiv a_{ij}^k,\ h_{ij}^k(t) \equiv t - \tau_{ij}^k$ and $a_{ij}^k,\ \tau_{ij}^k$ are constant) is considered in [8]. In particular, for the system

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij} x_j(t - \tau_{ij}), \quad i = 1, \dots, m, \tag{3}$$

where $\tau_{ij} \geqslant 0$, the following result holds (below, a_+ denotes the positive part of a, i.e., $a_+ = \max\{a, 0\}$).

Theorem 1 (*Corollary 4.3, [8]*). *Let*

$$0 < a_{ii}\tau_{ii} < 1 + 1/e, \quad i = 1, \dots, m$$

and let the $m \times m$ matrix H with components

$$h_{ij} = \begin{cases} \left(\frac{1-(a_{ii}\tau_{ii}-1/e)_+}{1+(a_{ii}\tau_{ii}-1/e)_+}\right) a_{ii}, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

$i,j = 1, \dots, m$ be a non-singular M-matrix. Then, system (3) is asymptotically stable for any selection of delays $\tau_{ij},\ i \neq j,\ i,j = 1, \dots, m$.

In [9], the system (3) is also considered and the following result derived.

Theorem 2 (*Theorem 1.3, [9]*). *Let*

$$0 \leqslant a_{ii}\tau_{ii} < 3/2, \quad i = 1, \dots, m$$

and let the matrix G with components

$$g_{ij} = \begin{cases} -\left(\frac{1+a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})/9}{1-a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})/9}\right)|a_{ij}|, & i \neq j, \\ a_{ii}, & i = j, \end{cases}$$

be a nonsingular M-matrix. Then, system (3) is asymptotically stable for any selection of delays $\tau_{ij},\ i \neq j,\ i,j = 1, \dots, m$.

In [10], the authors consider the non-autonomous system

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij}(t) x_j(h_{ij}(t)), \quad i = 1, \dots, m, \tag{4}$$

where $t \in [t_0, \infty),\ t_0 \in \mathbb{R},\ a_{ij}(t),\ h_{ij}(t)$ are continuous functions, $h_{ij}(t) \leqslant t$, and $h_{ij}(t)$ are monotone increasing functions such that $\lim_{t\to\infty} h_{ij}(t) = \infty,\ i,j = 1, \dots, m$.

Theorem 3 (*Theorem 2.2, [10]*). *Assume that, for $t \geqslant t_0$, there exist non-negative numbers $b_{ij},\ i,j = 1, \dots, m,\ i \neq j$ such that $|a_{ij}(t)| \leqslant b_{ij} a_{ii}(t),\ i,j = 1, \dots, m,\ i \neq j,\ a_{ii}(t) \geqslant 0$ and*

$$\int^{\infty} a_{ii}(s)ds = \infty, \quad d_i = \limsup_{t\to\infty} \int_{h_{ii}(t)}^{t} a_{ii}(s)ds < 3/2, \quad i = 1, \dots m.$$

Let $\tilde{B} = (\tilde{b}_{ij})_{i,j=1}^{m}$ be an $m \times m$ matrix with entries $\tilde{b}_{ii} = 1,\ i = 1, \dots, m$ and, for $i \neq j,\ i,j = 1, \dots, m$,

$$\tilde{b}_{ij} = \begin{cases} -\left(\frac{2+d_i^2}{2-d_i^2}\right) b_{ij}, & \text{if } d_i < 1, \\ -\left(\frac{1+2d_i}{3-2d_i}\right) b_{ij}, & \text{if } d_i \geqslant 1. \end{cases}$$

If $\tilde{B}$ is a nonsingular M-matrix, then system (4) is asymptotically stable.

Very interesting global asymptotic stability results were obtained for nonlinear systems of delay differential equations in the recent papers [15–17].

The aim of the paper is to extend Theorems 1–3 in the following directions. Instead of autonomous system (3) considered in Theorems 1 and 2, we consider non-autonomous system (1). Unlike of assumptions of Theorem 3, we remove inequalities $|a_{ij}(t)| \leqslant b_{ij} a_{ii}(t),\ i,j = 1, \dots, m,\ i \neq j$ and do not assume that $h_{ij}(t),\ i,j = 1, \dots, m$ are monotone increasing functions.

We will consider a more general system (1) and then, as a particular case, system (4) as well. We analyse systems with measurable parameters unlike the systems with continuous parameters investigated in [10].

In Theorems 1–3, all conditions are formulated in such a way that special matrices constructed here are non-singular M-matrices. We derive different stability conditions not assuming that a special matrix is an M-matrix and we show (in