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A parallel algorithm for generating ideal IC-colorings of cycles

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ABSTRACT

For a given graph G with the vertex set $V(G)$, a coloring $f : V(G) \to \mathbb{N}$ produces α where $\alpha = \sum_{u \in V(H)} f(u)$ for some connected subgraph H of G $\left(\sum_{u \in V(H)} f(u) = 0 \text{ if } V(H) = \varnothing\right)$. The coloring f is an IC-coloring of G if f produces each $\alpha \in \{0, 1, \ldots, S(f)\}$, where $S(f)$ is the maximum number that can be produced by f. The IC-index $M(G)$ of the graph G is the number $\max\{S(g)|g$ is an IC-coloring of $G\}$. An IC-coloring f is ideal if $S(f)$ is equal to the number of connected subgraph of G. In this paper, a sound and complete parallel algorithm based on the branch and bound technique is proposed to generate ideal IC-colorings of cycles, C_n. Experiments identified 118 ideal IC-colorings of C_n when $2 < n < 20$. Some cycles with particular length do not have any ideal IC-colorings while C_{18} has the maximal 51 ideal IC-colorings. No pattern appeared among cycles with ideal IC-colorings, regarding the length of cycles.

1. Introduction

For a given graph G with the edge set $E(G)$ and vertex set $V(G)$, a coloring $f : V(G) \to \mathbb{N}$ may *produce* α if $\alpha = \sum_{u \in V(H)} f(u)$ for some connected subgraph H of G $\left(\sum_{u \in V(H)} f(u) = 0 \text{ if } V(H) = \varnothing\right)$. The coloring f is an IC-*coloring* of G if f produces each $\alpha \in \{0, 1, \ldots, S(f)\}$, where $S(f)$ is the maximum number that can be produced by f. The IC-*index* $M(G)$ of the graph G is the number $\max\{S(g)|g$ is an IC-coloring of $G\}$. The number of connected subgraph of a graph G is the natural upper bound of IC-index as described by Salehi et al. [1]. An IC-coloring f of G is *maximal* if it is an IC-coloring of G when $S(f) = M(G)$. An IC-coloring f is *ideal* if $S(f)$ is equal to the number of connected induced subgraph of G. Saheli et al. [1] introduced the problem of finding IC-indices and IC-colorings of finite graphs. This problem may be considered as a derivative of the postage stamp problem in number theory [2–8]. Salehi et al. [1] have also studied the IC-indices of complete graphs, stars, double-starts, paths, cycles, and wheels. Shiue and Fu [9] obtained the IC-index of a complete bipartite graph, $K_{m,n}$, with $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$, for $2 \leqslant m \leqslant n$. Liu and Lee [10,11] investigated the IC-colorings and IC-indices of complete d-partite graphs and provided some properties and initial results.

This study focuses on finding ideal IC-colorings of a cycle C_n, with n nodes, which has $n(n-1)+1$ connected subgraphs, denoted as $I(C_n)$ and $M(C_n) \leqslant I(C_n)$, for any $n \geqslant 3$. Fig. 1 shows maximal IC-colorings of C_3, C_4, C_5, and C_6. Fink [4] presented a systematic way f to label cycles where f satisfies constraints of IC-colorings with $S(f) = n(n+1)/2$ that provides a lower bound of IC-index for cycles. Hence, the following inequality is obtained:

$$\frac{n(n+1)}{2} \leqslant M(C_n) \leqslant I(C_n), \quad \text{for any } n \geqslant 3.$$

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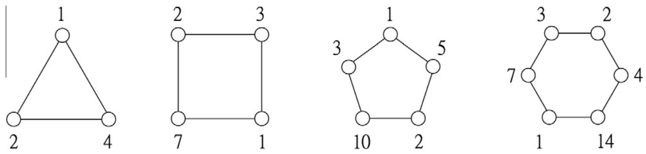

Fig. 1. Maximal IC-colorings of C_3, C_4, C_5, and C_6.

For IC-colorings illustrated in Fig. 1, the maximal IC-colorings are also ideal with $M(C_n) = I(C_n)$ for $3 \leqslant n \leq 6$. Without providing a systematic way to generate maximal IC-colorings of cycle, Salehi et al. [1] indicated that C_7 does not have any ideal IC-coloring. By exhaustive search, 13 maximal IC-colorings of C_7 are discovered. Fig. 2 shows Fink's [4] IC-coloring f with $S(f) = 28$ and one of the maximal IC-colorings of C_7 with $M(C_7) = 39$.

This preliminary investigation shows that while some maximal IC-colorings of cycles are ideal, others are not. Justified by the fact that the algorithm to generate maximal IC-colorings for cycles is not trivial, it is useful to know in advance whether a cycle with a particular length n has ideal IC-colorings. However, a simple unoptimized exhaustive search algorithm may not be practical in finding an ideal IC-coloring because it requires significant computational resources as the length of cycle increases. In this paper, an effective parallel algorithm based on the branch and bound (B&B) technique is proposed. Different scenarios of ideal IC-colorings of cycles with more than 2 vertices are investigated. The decision model (inclusion/exclusion principles) of the proposed algorithm is based on the two theorems described in Section 2 and these two theorems make the proposed algorithm sound and complete. In other words, the algorithm can generate all ideal IC-colorings of the cycle C_n. If it cannot, the ideal IC-coloring does not exist for C_n.

The rest of the paper is organized as follows. Section 2 describes all notations and theorems (with proof) used in this paper. Section 3 explains the decision model (inclusion/exclusion rules) used in the proposed algorithm. Section 4 is the section for the proposed algorithm followed by the experimental results in Section 5. Finally, the conclusions are presented in Section 6.

2. Notations and theorems

2.1. Notations

Vertex: $\langle f(v)\rangle$ indicates that $f(v)$ is produced by a single vertex v in a coloring f. For example, $\langle 4\rangle$ indicates that a vertex v exists where $f(v) = 4$.

Edge: $(f(v_1),f(v_2))$ indicates that there exists an edge between vertices v_1 and v_2. $(f(v_1),f(v_2))$ is equivalent to $(f(v_2),f(v_1))$. For instance, (1, 2) is equivalent to (2, 1); both indicate that two adjacent vertices v_1 and v_2 exist where $f(v_1) = 1$ and $f(v_2) = 2$.

Path: $(f(v_1),f(v_2),\ldots,f(v_m))$ represents a path (in a cycle) with m vertices $v_1, v_2, \ldots, v_m$ where $(f(v_i),f(v_{i+1}))$ holds for $1 \leqslant i \leqslant m-1$. Paths are also direction insensitive, similar to edges. $(f(v_1),f(v_2),\ldots,f(v_m))$ is equivalent to $(f(v_m),f(v_{m-1}),\ldots,f(v_1))$. For instance, path (1, 2, 3) is equivalent to (3, 2, 1) and the ideal IC-coloring for C_6 contains paths (4, 2, 3, 7) and (1, 7, 3), as shown in Fig. 1.

Derived Path: a path containing three or more vertices is called a derived path.

Cycle: $(f(v_1),f(v_2),\ldots,f(v_n))_r$ represents a cycle with n vertices $v_1, v_2, \ldots, v_n$ where both $(f(v_1),f(v_2),\ldots,f(v_n))$ and $(f(v_1),f(v_n))$ hold. For example, the ideal IC-coloring for C_6 contains $(1,7,3,2,4,14)_r$, as shown in Fig. 1.

Cycle Edge: $(f(v_1),f(v_2))_c$. An edge $(f(v_1),f(v_2))$ will be marked with a subscript 'c' if a path $(f(v_1), f(v_{i_1}),\ldots,f(v_{i_j}),f(v_2))$, $j \geqslant 1$, exists.

Path Permutation: $(f(v_1),f(v_2),\ldots,f(v_m))^*$ represents all paths that can be constructed by vertices $v_1, v_2, \ldots, v_m$. For instance, $(1,1,2)^*$ represents 2 paths: (1, 1, 2) and $(1,2,1)$; $(1,2,3)^*$ represents 3 paths:(1, 2, 3), (2, 1, 3), and (1, 3, 2). For an edge with two vertices v_1 and v_2, $(f(v_1),f(v_2)) = (f(v_1),f(v_2))^*$.

Tree Path: $\overline{(N_1,N_2,\ldots,N_m)}$ represents a decision tree path with m ordered nodes N_1 to N_m rooted at N_1. N_{i+1} is a child node of N_i for $1 \leqslant i \leqslant m-1$. A node can be a vertex or a path (including the cycle edge and the derived path).

2.2. Theorems

Theorem 2.1. *An ideal IC-coloring f for cycles does not contain two connected subgraphs H and I where* $\sum_{u\in V(H)} f(u) = \sum_{v\in V(I)} f(v)$ *and* $H \cap I = \varnothing$.

Proof. If an IC-coloring f for a cycle G is ideal, all G's $I(C_n)$ connected subgraphs must produce numbers from 1 to $I(C_n)$ distinctively. If two connected subgraphs produce the same α, the coloring f cannot be ideal, since at least one number exists, between 1 and $I(C_n)$, that cannot be produced. □