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On the classes of fractional order difference sequence spaces and their matrix transformations

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ABSTRACT

The main purpose of the present article is to introduce the classes of generalized fractional order difference sequence spaces $\ell_\infty(\Gamma, \Delta^{\tilde{\alpha}}, p)$, $c_0(\Gamma, \Delta^{\tilde{\alpha}}, p)$ and $c(\Gamma, \Delta^{\tilde{\alpha}}, p)$ by defining the fractional difference operator $\Delta^{\tilde{\alpha}} x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} x_{k+i}$, where $\tilde{\alpha}$ is a positive proper fraction and $k \in \mathbb{N} = \{1, 2, 3 \ldots.\}$. Results concerning the linearity and various topological properties of these spaces are established and also the alpha-, beta-, gamma- and N-duals of these spaces are obtained. The matrix transformations from these classes into Maddox spaces are also characterized. Throughout the article we use the notation $\Gamma(n)$ as the Gamma function of n, defined by an improper integral $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$, where $n \notin \{0, -1, -2, \ldots\}$ and $\Gamma(n+1) = n\Gamma(n)$.

1. Introduction, preliminaries and definitions

Let ω be the set of all sequences of real or complex numbers and ℓ_∞, c, c_0 be the set of all linear spaces that are bounded, convergent and null sequences $x = (x_k)$ with the complex terms, respectively, which are Banach spaces, normed by

$$\|x\|_\infty = \sup_k |x_k|,$$

where $k \in \mathbb{N} = \{1, 2, 3 \ldots\}$, the set of positive integers. The notion of difference sequence space was first introduced by Kızmaz [1] by defining the sequence space

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}, \tag{1.1}$$

for $X = \ell_\infty$, c and c_0, where $\Delta x = (x_k - x_{k+1})$. Later on the above idea was generalized by Et and Çolak [2] as follows:

$$X(\Delta^r) = \{x = (x_k) : \Delta^r x \in X\}, \tag{1.2}$$

where $r \in \mathbb{N}$ and $\Delta^r x = \left(\sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} r \\ i \end{pmatrix} x_{k+i}\right)$. It was observed that the above spaces are Banach spaces with respect to the norm

$$\|x\|_\Delta = \sum_{i=1}^{r} |x_i| + \|\Delta^r x\|_\infty. \tag{1.3}$$

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Subsequently, this concept was studied and extended by Bektas et al. [3], Bektas and Et [4], Et [5], Et and Nuray [6] and many others. The main focus of this work is to unify most of the difference sequence spaces defined earlier and extend these results to the fractional case.

For a positive proper fraction $\tilde{\alpha}$ and a bounded sequence of positive reals (p_k), we introduce the fractional order difference sequence spaces $\ell_\infty(\Gamma, \Delta^{\tilde{\alpha}}, p)$, $c_0(\Gamma, \Delta^{\tilde{\alpha}}, p)$ and $c(\Gamma, \Delta^{\tilde{\alpha}}, p)$, defined by

$$\ell_\infty(\Gamma, \Delta^{\tilde{\alpha}}, p) = \left\{ x = (x_k) \in \omega : \sup_k |\Delta^{\tilde{\alpha}} x_k|^{p_k} < \infty \right\},$$
$$c_0(\Gamma, \Delta^{\tilde{\alpha}}, p) = \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} |\Delta^{\tilde{\alpha}} x_k|^{p_k} = 0 \right\},$$
$$c(\Gamma, \Delta^{\tilde{\alpha}}, p) = \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} |\Delta^{\tilde{\alpha}} x_k - L|^{p_k} = 0, \quad \text{for some } L \in \mathbb{C} \right\},$$

where $\Delta^{\tilde{\alpha}}$ is called the fractional difference operator and defined by

$$\Delta^{\tilde{\alpha}} x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} x_{k+i}. \tag{1.4}$$

For instance,

- $\Delta^{\frac{1}{2}} x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} \ldots$
- $\Delta^{-\frac{1}{2}} x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} \ldots$
- $\Delta^{\frac{1}{3}} x_k = x_k - \frac{1}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{5}{81}x_{k+3} - \frac{10}{243}x_{k+4} - \frac{22}{729}x_{k+5} \ldots$
- $\Delta^{\frac{2}{3}} x_k = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \frac{14}{729}x_{k+5} \ldots$

For our investigation, throughout it is being assumed that the summation in (1.4) is convergent. If $\tilde{\alpha}$ is a nonnegative integer, then the infinite sum defined in (1.4) reduces to a finite sum, i.e. $\Delta^{\tilde{\alpha}} x_k = \sum_{i=0}^{\tilde{\alpha}} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} x_{k+i}$.

In particular, the difference operator $\Delta^{\tilde{\alpha}}$ includes the following special cases:

(i) For $\tilde{\alpha} = 1$ and $p_k = 1$, this operator generalizes the difference operator introduced by Kızmaz [1].
(ii) For $\tilde{\alpha} = m \in \mathbb{N}$ and $p_k = 1$, this operator generalizes the difference operator defined by Et and Çolak [2].
(iii) For $\tilde{\alpha} = 1$, this operator generalizes the difference operator considered by Ahmad and Mursaleen [7].
(iv) For $\tilde{\alpha} = m \in \mathbb{N}$, this operator generalizes the difference operator considered by Et and Basarir [8].

2. The fractional difference operator $\Delta^{\tilde{\alpha}}$

For last three decades, eminent mathematicians such as Kızmaz [1], Et [5], Et and Çolak [2], Altay and Başar [9,10], Dutta and Baliarsingh [10,11] and many others have introduced, investigated and enriched the existing sequence spaces through the difference operator Δ^m where m is a positive integer. In fact, the development of the different sequence spaces has been taken place due to the introduction of several new modern techniques in functional analysis involving topological structures, dual spaces, matrix mappings etc (see [12–31]). The main objective of this article is to extend these results to the fractional difference case. Now, in this section we investigate and discuss some interesting properties of the fractional difference operator $\Delta^{\tilde{\alpha}}$.

Theorem 1. *For a proper fraction $\tilde{\alpha}$, the operator $\Delta^{\tilde{\alpha}} : X \to Y$, defined by $\Delta^{\tilde{\alpha}} x = (\Delta^{\tilde{\alpha}} x_k)$, is linear, where $X, Y \subset \omega$ and $\Delta^{\tilde{\alpha}} x_k$ is defined by (1.4).*

Proof. The proof of this theorem is trivial, hence omitted. □

Theorem 2. *For any two proper fractions α_1 and α_2*

(i) $\Delta^{\alpha_1}(\Delta^{\alpha_2} x_k) = \Delta^{\alpha_2}(\Delta^{\alpha_1} x_k) = \Delta^{\alpha_1+\alpha_2}(x_k)$,
(ii) $\Delta^{\alpha}(\Delta^{-\alpha} x_k) = \Delta^{-\alpha}(\Delta^{\alpha} x_k) = x_k$,
(iii) *If* $\alpha_1 + \alpha_2 = 1$, *then* $\Delta^{\alpha_1}(\Delta^{\alpha_2} x_k) = \Delta^{\alpha_1+\alpha_2}(x_k) = \Delta x_k = x_k - x_{k+1}$.

Proof

(i) For two proper fractions α_1, α_2 and by Theorem 1, we have