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A new iterative method for solving linear Fredholm integral equations using the least squares method

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ABSTRACT

In this paper, a new iterative method is proposed for solving linear integral equations. This method is based on the *LSQR* method, an algorithm for sparse linear equations and sparse least squares, reducing the solution of linear integral equations to the solution of a bidiagonal linear system of algebraic equations. A simple recurrence formula is presented for generating the sequence of approximate solutions. Some theoretical properties and error analysis of the new method are discussed. Although the new method can be used for solving the ill-posed first kind integral equations independently, combining of the new method with the method of regularization is presented to solve this kind of integral equations. Also the perturbing effect of the first kind integral equations is analyzed. Some properties and convergence theorem are proposed. Finally, some numerical examples are presented to show the efficiency of the new method.

1. Introduction

Many applications require the solution of linear integral equations of the first or second kinds that can be written in the general form

$$\lambda u(t) - \int_\Omega K(t,s)u(s)ds = f(t), \quad \Omega = [a,b], \tag{1}$$

where f is known function, K is continuous kernel and u is an unknown function to be determined. Eq. (1) is reduced to a first kind of integral equation if $\lambda = 0$. Integral equations arise naturally in applications, in many areas of mathematics, science and technology and have been studied extensively both at the theoretical and practical level. Many problems of mathematical physics can be stated in the form of integral equations. There is almost no area of applied mathematics and mathematical physics where integral equations do not play a role [2,9,19].

A variety of analytic and numerical methods have been used to handle integral equations including the series solution method [11], Adomian decomposition method [1,17,19] and its modification [18,19], the variational iteration method(VIM) [8,19], the homotopy perturbation method [7,19], Nyström [2,3,13] method and so on. Projection methods are another most important classes of numerical methods used for solving integral equations [13]. In these methods, a sequence of finite-dimensional approximating subspaces V_n of some complete function space V, usually $C(\Omega)$ or $L^2(\Omega)$, are chosen. Then a function $u_n \in V_n$ is sought, which can be written as

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$$u_n(t) = \sum_{j=1}^{\kappa_n} c_j \phi_j(t), \quad t \in \Omega,$$

where $\{\phi_1, \phi_2, \ldots, \phi_{\kappa_n}\}$ is a basis of V_n. This is substituted into the given integral equation and the coefficients $\{c_1, c_2, \ldots, c_{\kappa_n}\}$ are determined by forcing the equation to be almost exact in some sense. In the projection methods, the basis of V_n is chosen in different ways, usually prefabricate. Examples, are piecewise linear approximating function, trigonometric polynomials [3] and so on.

The bulk of the work on iterative methods for the solution of linear integral equations is devoted to equations which have unique solutions whereas, linear integral equations of the first and second kinds that have non-unique solutions or that have no solution at all arise in many settings. So, there are a number of cases in which one would like to find the solution of minimal norm to a non-uniquely solvable integral equations, or to seek least-squares solutions when the integral equation does not have a solution in the classical sense [10]. Accordingly, one of the purpose of the present paper is to investigate best approximate solutions, i.e., solutions in the sense of least square. In addition, we use our approach to solve the first kind linear integral equation without transforming it to a second kind equation and we do not stipulate that the equation is uniquely solvable since we seek solutions in the least-squares sense.

In this paper, we extend the applicability of the *LSQR* method [14] to linear operator equations $\mathcal{L}u = f$ including the special cases where $\mathcal{L}$ is a integral operator of the first or second kind. This method reduces the solution of linear operator equations to the solution of a bidiagonal linear system of algebraic equations. It generates a sequence of approximations $\{u_k\}$ such that the residual norm $\|r_k\|$ decreases monotonically, where $r_k = f - \mathcal{L}u_k$. Since the *LSQR* method [14] is based on the bidiagonalization process of Golub and Kahan [6], we first present the bidiagonalization process for the linear operator $\mathcal{L}$. This process generates two sets of functions namely $\psi_1, \psi_2, \ldots$ and $\phi_1, \phi_2, \ldots$ such that

$$\langle \psi_i, \psi_j \rangle = \delta_{ij} \quad \text{and} \quad \langle \phi_i, \phi_j \rangle = \delta_{ij}, \quad i,j = 1,2,\ldots,$$

where δ_{ij} is

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$$

and $\langle , \rangle$ is an inner product which is defined in the following section.

We use the generated functions $\psi_1, \psi_2, \ldots$ as basis functions to approximate the solution of linear integral Eq. (1). The new method has certain advantages, namely it is an iterative method, we do not need to store the basis functions, and the approximate solutions and residuals are cheaply computed at each stage of the algorithm because they are updated with short-term recurrences. At each stage of the presented algorithm, computation of some definite integrals are needed which can be carried out by an appropriate numerical integration. Also we apply the new presented method to solve the first kind Fredholm integral equation and its perturbed form.

The outline of the paper is as follows. In Section 2, we give a short overview of the *LSQR* method and its properties. In Section 3, we present a new iterative method and its error analysis. Combining of the new method with the method of regularization for solving ill-posed first kind Fredholm integral equations and perturbation analysis are given in Section 4. In Section 5, some numerical examples are presented to show the efficiency of the new method. Finally, we make some concluding remarks in Section 6.

2. A brief description of the LSQR method

One of the well known iterative methods, for solving square and rectangular linear system s of equations, is the *LSQR* method [14]. Hence in this section, we recall some fundamental properties of the *LSQR* method. Consider the linear system of equations

$$Ax = b, \tag{2}$$

where A is a large $n \times m$ complex matrix, $b \in \mathbb{C}^m$ and $x \in \mathbb{C}^n$. The *LSQR* method uses a bidiagonal process to reduce the coefficient matrix A to the lower bidiagonal form. The bidiagonal process can be described as follows. The bidiagonalization with starting vector b is initialized with

$$\beta_1 u_1 = b, \quad \alpha_1 v_1 = A^H u_1,$$

$$\left.\begin{aligned} \beta_{i+1} u_{i+1} &= A v_i - \alpha_i u_i \\ \alpha_{i+1} v_{i+1} &= A^H u_{i+1} - \beta_{i+1} v_i \end{aligned}\right\}, \quad i = 1,2,\ldots, \tag{3}$$

where $u_i, v_i \in \mathbb{C}^n$ and A^H denotes the conjugate and transpose of A. The scalars $\alpha_i \geqslant 0$ and $\beta_i \geqslant 0$ are chosen so that $\|u_i\|_2 = \|v_i\|_2 = 1$. With the definitions

$$U_k = [u_1, u_2, \ldots, u_k] \in \mathbb{C}^{n \times k}, \quad V_k = [v_1, v_2, \ldots, v_k] \in \mathbb{C}^{m \times k},$$