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Performance comparison of numerical inversion methods for Laplace and Hankel integral transforms in engineering problems

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A B S T R A C T

Different methods for the numerical evaluations of the inverse Laplace and inverse of joint Laplace–Hankel integral transforms are applied to solve a wide range of initial-boundary value problems often arising in engineering and applied mathematics. The aim of the paper is to present a performance comparison among different numerical methods when they are applied to transformed functions related to actual engineering problems found in the literature. Most of our selected test functions have been found in the solution of boundary value problems of applied mechanics such as those related to transient responses of isotropic and transversely isotropic half-space to concentrated impulse or those related to visco-elastic wave motion in layered media. These classes of test functions are frequently encountered in similar problems such as those in boundary element or boundary integral equations, theoretical seismology, soil–structure-interaction in time domain and so on. Therefore, their behavior with different numerical inversion algorithms could make a useful guide to a precise choice of more suitable inversion method to be used in similar problems. Some different methods are also investigated in detail and compared for the inversion of the joint Hankel–Laplace transforms, where more sophisticated integrand functions are encountered. It is shown that Durbin, Crump, D'Amore, Fixed-Talbot, Gaver–Whyn–Rho (GWR), and Direct Integration methods have excellent performance and produce good results when applied to the same problems. On the contrary, Gaver–Stehfest and Piessens methods furnish results not very reliable for almost all classes of transformed functions and they seem good only for "*simple*" transformed functions. Particularly the performance of GWR algorithm is very good even for transformed functions with infinite number of singularities, where the other methods fail. In addition, in case of double integral transforms, only the Fixed-Talbot, Durbin and Weeks methods are recommended.

1. Introduction

We denote by $\bar{F}(p)$ the one sided Laplace transform of a function $f(t)$ defined for all $t \in (0, \infty)$ as:

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$$\bar{F}(p) = \int_0^\infty f(t)e^{-pt}dt \tag{1}$$

where $\bar{F}(p)$ is a complex function defined for those p where the right hand side in (1) is convergent. The necessary conditions for the existence of (1) are

$$\mathrm{Re}[p] > \sigma_0 \tag{2}$$

$$|f(t)| < o(e^{-\sigma_0 t}) \quad \text{as } t \to \infty \tag{3}$$

where σ_0 is the abscissa of convergence of f and must be chosen greater than the real parts of all singularities of $\bar{F}(p)$. If conditions (2) and (3) hold, the function $\bar{F}(p)$ in Eq. (1) is analytic in half-plane $\mathrm{Re}[p] > \sigma_0$ and the inversion of integral (1) is [7]

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \bar{F}(p)e^{pt}dp, \tag{4}$$

in which σ is an arbitrary real number such that $\sigma > \sigma_0$.

The integral transforms are powerful and thus common tools for analytical and numerical investigations in a wide range of engineering initial-boundary value problems and applied mathematics [4,26,27,30,41,64], such as wave propagation, geophysics, heat transfer, viscoelasticity, earthquake engineering, soil dynamics, electrochemistry and hydrology. Another interesting application of Laplace transform is the analysis of data deriving from industrial applications in chemistry, physics and medicine acquired by NMR spectroscopy where the Laplace transform describes the exponential decay of the NMR signal [5,33]. In each of these problems, researchers have to take into account the difficulties lurking in the numerical inversion of the transformed function in order to compute the solution in the real domain.

The scientific and applicative interest of Laplace transform inversion has been shown by more than 1000 publications in the literature of the last century. In the applications, it is almost impossible to know the inverse solution in closed form as may be found in the table of integral transforms [2,19]. Thus, a numerical inversion is needed. About one hundred algorithms concerning numerical inversion of Laplace transform were published over the last years and only a few of them can be effectively used in mathematical software [10]. Because of the strong difference and critical aspects underlying the numerical inversion approaches, the real inversion problem is distinguished from the complex one in the applications related to the computation of the Laplace inverse function. In most of the applications, the methods and algorithms of numerical *real* inversion are used, however, because of the well known ill-posedness of the problem of real inversion, they suffer a significant ill-conditioning with subsequent amplification of the errors. As a consequence, the related software, sometimes cannot meet the requirements of reliability and robustness needed in the applications. For example, in the analysis of experimental data, the function to be inverted is known only through a finite number of samples, which means the inversion procedure can be realized after a preprocessing step, in which a fitting model of the data set mimics the behavior of the unknown Laplace transform generating function [9]. In every case, the user's experience determines the choice, more or less adequate, of the algorithm with which to solve the problem at hand. One of the early comparative and systematic studies among different methods of numerical Laplace transform inversion, was done by Davis and Martin [13], in which 14 methods were analyzed and tested for some transformed functions, whose inversion are exactly known. Two other comparative studies are reported by Naryanan and Beskos [43] and Duffy [16]. In the former, 8 different methods for Laplace transform inversion were investigated to evaluate the solution of some boundary value problems of applied mechanics, which were originally solved with the use of Laplace transform in conjunction with some numerical techniques for the numerical solution of partial differential equations. Duffy [16] tested three software packages for numerical inversion of Laplace transform. The interested reader may find a wider bibliography in Belman [6] and Cohen [7]. In a wide range of engineering problems, Laplace transform is used in conjunction with Hankel integral transform [20,21,29,44,48,50], where the Hankel integral transform of order v of function $f(r)$, $r \in (0, \infty)$, is defined as [53]

$$\tilde{f}^v(\xi) = \int_0^\infty rf(r)J_v(\xi r)dr, \tag{5}$$

where $J_v(.)$ is the Bessel functions of the first kind and order v. Then the inverse Hankel integral transform is expressed as:

$$f(r) = \int_0^\infty \xi\tilde{f}^v(\xi)J_v(\xi r)d\xi \tag{6}$$

The necessary condition for the Hankel integral transform to be existed is that

$$\lim_{\beta\to\infty}\int_0^\beta \sqrt{r}|f(r)|dr < \infty \tag{7}$$

The joint Hankel–Laplace transform is a common tool for three-dimensional transient wave propagation in both elastic and thermoelastic boundary value problems, specially in cylindrical coordinate system [26,27,50,58,49]. Sometimes, by separation of two integral transforms that happens with the use of the so-called Cagniard–De Hoop method [3], it is possible to invert one of the integral transform in analytical form. In contrast to the Laplace integral transform, the Hankel transform is self-reciprocal and its inversion is always a well-posed problem. In simple cases, the Hankel inversion may also be evaluated