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Golden section, Fibonacci sequence and the time invariant Kalman and Lainiotis filters

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ABSTRACT

We consider the discrete time Kalman and Lainiotis filters for multidimensional stochastic dynamic systems and investigate the relation between the golden section, the Fibonacci sequence and the parameters of the filters. Necessary and sufficient conditions for the existence of this relation are obtained through the associated Riccati equations. A conditional relation between the golden section and the steady state Kalman and Lainiotis filters is derived. A Finite Impulse Response (FIR) implementation of the steady state filters is proposed, where the coefficients of the steady state filter are related to the golden section. Finally, the relation between the Fibonacci numbers and the discrete time Lainiotis filter for multidimensional models is investigated.

1. Introduction

Although the connection between the golden section with nature, arts and architecture is known for centuries, there is presently a huge interest of modern sciences in these classical theories. Particularly, researchers in computer science (measurement theory, graph theory and communication systems [20–22]) and cryptography [18] exhibit a substantial interest in these classical theories and use them in order to model phenomena in their field. The above are only a few applications of the golden section that imply a new mathematical direction which is the creation of a fascinating and beautiful subject of the "Mathematics of Harmony" [9,19] and the references therein. The relation between the discrete time Kalman filter/Lainiotis filter and the golden section is described for scalar systems in [6,8,10,11] and for special multidimensional systems in [10,12]; in [10] two cases are examined (i) the noise covariances, transition and measurement matrices are equal to the identity matrix, (ii) the output matrix is the identity matrix, and in [12] the elements of the steady state covariance and gain matrices are functions of the golden ratio.

This paper examines the situation in which the parameters of the Kalman filter [2,15] and Lainiotis filter [16] for more general multidimensional stochastic dynamic systems related to the golden section and the Fibonacci sequence, extending the results in [6,10]. Concerning the novelty of the paper, we mention that: (i) we generalize the results obtained in [10] formulating specific assumptions on the transition matrix and describing the conditional relation between the golden section and Kalman and Lainiotis filters for more general multidimensional systems and (ii) we investigate the theoretical

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properties of the transition matrix and its relation with the covariance matrices in order to be able to been guaranteed a relation between the golden section and Kalman and Lainiotis filters.

The paper is organized as follows: In Section 2, a review of the discrete time Kalman and Lainiotis filters and a necessary review of the matrix analysis are presented. In Section 3, the multidimensional stochastic dynamic system is considered. Necessary and sufficient conditions for the existence of the relation between the discrete time Kalman filter/Lainiotis filter and the golden section are obtained through the associated Riccati equations. Also a location of eigenvalues and the spectral radius of the transition matrix of the system is presented. In Section 4, the relation between the steady state Kalman filter/Lainiotis filter and the golden section is described and a Finite Impulse Response (FIR) implementation of the steady state filters is proposed, where the coefficients of the filter are related to the golden section. In Section 5, the relation between the Fibonacci numbers and the discrete time Lainiotis filter is presented. Finally, Section 6 is devoted to conclusions.

2. Notation and preliminaries

2.1. Golden section, Golden ratio and Fibonacci sequence

The terms "golden section" and "golden ratio" and as a concept has a long history in mathematics, see e.g. [6] and the references therein. *Golden section*, α, is called the positive root of the equation $\lambda^2 + \lambda - 1 = 0$, which is equal to

$$\alpha = \frac{-1+\sqrt{5}}{2} \approx 0.618 \tag{1}$$

and *golden ratio*, ϕ, is the reciprocal of the golden section, which is given:

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \tag{2}$$

The relations between the golden section and the golden ratio are derived by (1), (2) and are given below:

$$\phi = \frac{1}{\alpha} = 1 + \alpha \tag{3}$$

For each $v \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ from (3) arises

$$\frac{1}{1+\phi^v} = \frac{\alpha^v}{1+\alpha^v}. \tag{4}$$

The Fibonacci sequence is defined by recurrence by taking each subsequent number as the sum of the two previous ones

$$f_{v+2} = f_{v+1} + f_v, \quad v \in \mathbb{Z}_+ = \{1, 2, \ldots\}, \text{ with } f_1 = 0, \quad f_2 = 1. \tag{5}$$

Thus, the sequence of Fibonacci numbers is $\{0, 1, 1, 2, 3, 5, 8, \ldots, \}$, [1,17]. It is well-known that the Fibonacci sequence satisfies the limit properties [1,8]

$$\lim_{v\to\infty} \frac{f_v}{f_{v+1}} = \alpha \text{ and } \lim_{v\to\infty} \frac{f_{v+2}}{f_{v+1}} = \phi, \tag{6}$$

which provide the relation between the Fibonacci sequence on the one hand and the golden section and the golden ratio on the other hand.

2.2. Stochastic dynamic system

Consider the time invariant stochastic dynamic system described by the following state space equations

$$\begin{aligned} x_{k+1} &= Fx_k + w_k \\ z_k &= Hx_k + v_k \end{aligned} \tag{7}$$

for $k = 0, 1, \ldots$, where x_k is $n \times 1$ state vector at time k, z_k is $m \times 1$ measurement vector, F is $n \times n$ transition matrix, H is $m \times n$ output matrix, $\{w_k\}$, $\{v_k\}$ are the plant noise and the measurement noise process, respectively. These processes are assumed to be Gaussian, zero-mean, white and uncorrelated random processes with Q, R be $n \times n$ plant noise and $m \times m$ measurement noise covariance matrices, respectively. The vector x_0 is a Gaussian random process with mean $\bar{x}_0$ and covariance P_0. In the sequel, consider that Q, R, P_0 are positive definite matrices. Also, x_0, $\{w_k\}$ and $\{v_k\}$ are independent.

The filtering problem is to produce an estimate at time L of the state vector using measurements till time L, i.e., the aim is to use the measurements set $\{z_1, z_2, \ldots, z_L\}$ in order to calculate an estimate value $x_{L/L}$ of the state vector x_L.