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Limit cycles of cubic polynomial differential systems with rational first integrals of degree 2

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ABSTRACT

The main goal of this paper is to study the maximum number of limit cycles that bifurcate from the period annulus of the cubic centers that have a rational first integral of degree 2 when they are perturbed inside the class of all cubic polynomial differential systems using the averaging theory. The computations of this work have been made with Mathematica and Maple.

1. Introduction and statement of the main results

One of the main problems in the qualitative theory of real planar differential systems is the determination of their limit cycles. There are several methods for computing the number of limit cycles bifurcating from the periodic orbits of a center [1,2]. Most of the methods are based on the Poincaré return map, the Poincaré–Melnikov integrals (see for instance [8,9]), the Abelian integrals, and the averaging theory of first order. In fact in the plane the last three methods are essentially equivalent. The first two methods only give the number of the periodic orbits of the unperturbed system that become limit cycles when the system is perturbed. The averaging method and method presented in [5] using the inverse integrating factor also give the shape of the bifurcated limit cycles. For a general overview of some of these mentioned tools see the book [7].

The study of the number of limit cycles of a polynomial differential system is mainly motivated by the 16th Hilbert problem, which together with the Riemann conjecture are the two problems of the famous list of 23 problems of Hilbert which remains open. See for more details [10,16].

The problem of studying the bifurcation of limit cycles from the periodic orbits of a center of a polynomial differential system of degree 2 when this system is perturbed inside the class of all polynomial differential systems of degree 2 has been studied intensively during these last 20 years, see for instance the books [4,18], and the hundreds of references quoted there, and in particular the references [6,13,19]. There are few works trying to study this problem for cubic polynomial differential systems. Our objective will be to study this problem for the cubic polynomial differential systems having a rational first integral of degree 2.

The classification of all cubic polynomial differential systems having a center at the origin and a rational first integral of degree 2 can be found in [14]. Now we summarize this classification in six families of cubic polynomial differential systems that we denote by P_k for $k = 1, 2, \dots, 6$.

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The class P_1 is represented by system

$$\dot{x} = 2y(\alpha^2 + \beta + 2\alpha x + x^2),$$
$$\dot{y} = -2((\alpha^2 + \beta)x + \alpha x^2 - \alpha y^2 - xy^2),$$

with $\beta < 0$ and $\alpha^2 + \beta \neq 0$.

The class P_2 is obtained translating the other center of P_1 to origin. It is described by system

$$\dot{x} = 2\alpha^{-2}y(\beta^2 - 2\alpha\beta x + \alpha^2(\beta + x^2)),$$
$$\dot{y} = -2(\alpha^2 x - \alpha x^2 - \alpha^{-2}\beta y^2 + x(\beta + y^2)).$$

The class P_3 is given by system

$$\dot{x} = x(1 + \alpha^2 + 2x + x^2 - y^2),$$
$$\dot{y} = y(-1 - \alpha^2 - 2y + x^2 - y^2),$$

with $\alpha \neq 0$.

The class P_4 is represented by system

$$\dot{x} = y(x^2 + \alpha), \quad \dot{y} = x(y^2 + \beta),$$

with $\alpha\beta < 0$.

The class P_5 is provided by system

$$\dot{x} = x(\beta^2 + d^2 + 2(\beta + \gamma d)x + (\gamma^2 + 1)x^2 - y^2),$$
$$\dot{y} = y(-\beta^2 - d^2 - 2dy + (\gamma^2 + 1)x^2 - y^2),$$

with $d = 0$ or $d = 1$. If $d = 0$ then $\beta\gamma \neq 0$ and if $d = 1$ then $\beta(\beta\gamma - 1) \neq 0$.

Finally the class P_6 is obtained taking $\beta = 0$ in P_1. Thus, we get the system

$$\dot{x} = 2y(x + \alpha)^2,$$
$$\dot{y} = -2(x + \alpha)(\alpha x - y^2),$$

with $\alpha \neq 0$.

In [12] the authors studied the cubic polynomial differential systems having a rational first integral of degree 2 whose phase portraits correspond to the phase portraits P_1, P_3 and P_4 of Fig. 1. These systems were denoted in [12] by (A), (B) and (C). They also proved that all the centers of these systems are reversible and isochronous, see [12, p. 314]. Their main result provides upper bounds $\mathcal{M}$ for the maximum number of limit cycles of systems P_1, P_3 and P_4 when they are perturbed

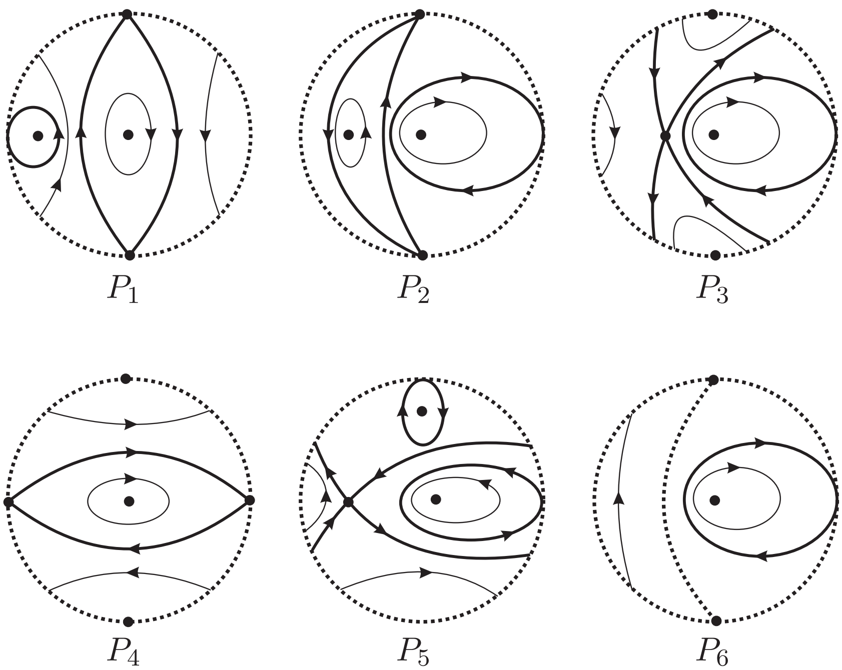

Fig. 1. Phase portraits in the Poincaré disc of the cubic polynomial vector fields with a center at the origin and a rational first integral of degree 2.