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Pricing forward-start variance swaps with stochastic volatility

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ABSTRACT

In this paper, a general approach is presented to price forward-start variance swaps with discrete sampling times, based on the Heston (1993)'s two-factor stochastic volatility model. Using this approach we work out two analytical closed-form formulae for the price of forward-start variance swap with the realized variance being defined by the actual-return realized variance and the log-return realized variance, respectively. The main features of this new approach and the developed formulae as a result include: (1) treating the pricing problem of variance swaps with different definitions of discretely-sampled realized variance in a highly unified way; (2) easily obtaining analytical closed-form solutions for forward-start variance swaps with two popularly-used definitions of discretely-sampled realized variance; (3) enabling the investigation of some important properties of the forward-start variance swaps, utilizing the elegant and simple form of the obtained solutions. With these advantages, we believe that the approach can be applied to price variance swaps based on other stochastic volatility models as well.

1. Introduction

Variance swap appears to be an actively pursued research topic in the last decades, particularly because of the recent volatilities experienced by almost all financial markets in the world even since the sub-prime crisis started in 2007. Prior to this work, studies on variance swaps were primarily limited to the special kind of instantaneous-start variance swaps, whereas it is quite often that those traded in markets or even some over-the-counter ones are of a forward-start nature, characterized by the starting time of the sampling period being a future date, when the market volatility is another unknown variable, in addition to all the unknown volatilities at the sampling points within the sampling period. This paper addresses this forward-start feature by proposing a new approach.

Variance swaps are essentially forward contracts on annualized realized variance that provide an easy way for investors to trade future realized variance against the current implied variance. Forward-start variance swaps are a kind of variance swaps whose annualized realized variance is measured between two future dates T_s and T_e, where $0 < T_s < T_e$, with $t = 0$ being the current time. Even though forward-start variance swaps seem to be a simple and natural extension to the standard variance swaps, whose annualized realized variance is calculated all the time between now and a future time T_e, the introduction of the forward-start feature can increase the flexibility of variance swaps in hedging risk, and hence greatly promote the trading volumes of variance swaps. For example, the standardized forward-start variance swaps, named as variance futures, have already been actively traded in some stock exchanges. Chicago Board Options Exchange (CBOE) launched

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3-month and 12-month variance futures on S&P 500 in May 2004 and March 2006, respectively. New York Stock Exchange (NYSE) Euronext also started to offer variance futures on FTSE 100, CAC 40 and AEX indices in September 2006. All those listed variance futures are nothing but forward-start variance swaps.

Proving the effective exposure to volatility, volatility and variance swaps have been actively traded over the last few years. A report[1] from CBOE indicates that "a recent estimate from *risk* magazine placed the daily volume in variance swaps on the major equity-indices to be US$5M vega (or dollar volatility risk per percentage point change in volatility) in the OTC markets. Furthermore, variance trading has roughly doubled every year for the past few years". Broadie and Jain [4] even estimated that daily trading volume on indices was in the region of $30 millions to $35 millions notional. The interest in trading volatility-based financial derivatives, such as variance swaps, seems to be still strongly growing among hedge funds and proprietary desks as [24] pointed out. It can be imagined that recent market turmoil due to the US subprime crisis would further intensify the trading of volatility-based financial derivatives, and thus greatly promote research in this area.

With the rapid growth of trading volumes in variance swaps, considerable research interest has been attracted to develop appropriate valuation approaches and trading strategies. The most noticeable ones were proposed by Carr and Madan [10] and Demeterfi et al. [13]. They have shown how to theoretically replicate a variance swap by a portfolio of standard options. Without requiring to specify the function of volatility process, their models and analytical formulae are indeed very attractive. However, as pointed out by Carr and Corso [7], the replication strategy has a drawback that the sampling time of a variance swap has to be assumed continuous rather than discrete; such an assumption implies that the results obtained from a continuous model can only be viewed as an approximation for the actual cases in financial practice, in which all contracts are written with the realized variance being evaluated on a set of discrete sampling points. Of course, one may argue that if the sampling period of the discretely-sampled realized variance is sufficiently small, continuous sampling models will provide good approximations. Using this argument, many researchers have adopted various stochastic volatility models and presented different pricing formulae for variance swaps as well as other volatility-based products. Typical examples of these studies include Grunbichler and Longstaff [19], Brockhaus and Long [5], Javaheri et al. [23], Howison et al. [21], Swishchuk [34], Carr and Lee [8,9], Elliott et al. [16] and Sepp [31], etc. Most these stochastic volatility models, again, share two common features and limitations: (1) their results are based on the assumption that the realized variance is defined by a continuously-sampling approximation; and (2) their discussions and results are only applicable for the instantaneous starting variance swaps, without considering the forward-start feature. However, in practice the realized variance of a variance swap is always discretely sampled, and most of the traded variance swaps are forward-start, rather than instantaneous-start, contracts. As a result, one naturally wonders how much bias the continuously-sampling approximation results in and how the forward-start feature effects the price of a variance swap.

The first issue has been studied recently. A typical example is the paper published by Little and Pant [25], who showed how to price a discretely-sampled variance swap using the finite-difference method in an extended Black–Scholes framework with the local volatility being assumed to be a known function of time and spot price of the underlying asset. Windcliff [36] also explored a numerical algorithm to evaluate discretely-sampled volatility derivatives using numerical partial-integro differential equation approach, under local volatility models, jump-diffusion models and models with transaction cost being taken into consideration. Although these numerical methods evaluate variance swaps based on discretely-sampled realized variance and achieve reasonably high accuracy, local volatility models are not really suitable for pricing volatility derivatives as a presumed known volatility surface has defeated the whole purpose of pricing a derivative contract whose payoff depends up on at least one unknown variable at the future payoff time. To remedy this drawback, Little and Pant [25] and Windcliff et al. [36] pointed out, respectively, that for better pricing and hedging general variance swaps one needs to adopt an appropriate model that incorporates the stochastic volatility characteristics observed in financial markets. In fact, it is worth mentioning that stochastic models have not only been used in quantitative finance in terms of pricing financial derivatives, but also gained popularity in other social science field concerning large-scale dataset analysis, such as studies in human culture (see [17,28,26,27]).

There have been several very recent studies, using stochastic volatility models to price discretely-sampled variance swaps. Broadie and Jain [6] presented a closed-form solution for volatility as well as variance swaps with discrete sampling. Their solution approach is primarily based on integrating the underlying stochastic processes directly. It appears that their approach can only be used when the realized variance is defined in such a particular form that the stochastic processes assumed for the underlying can happen to be exactly the same as that defined in the calculation of the realized variance. Alternatively, Zhu and Lian [39,41] presented a completely different approach, by analytically solving the associated PDEs, to obtain closed-form formulae for variance swaps based on the discretely-sampled realized variance. Unlike Broadie and Jain's [6] approach, Zhu and Lian's [39,41] approach of solving the governing PDE system directly produces two much simpler formulae and is more versatile in terms of dealing with different forms of realized variance.

However, up to now, none in literature has taken the forward-start feature into consideration which is usually imbedded in most of traded variance swaps, in the context of stochastic volatility and discretely-sampled realized variance. In this paper, we present an approach to price discretely-sampled forward-start variance swaps based on Heston's two-factor stochastic volatility model. In this way, the nature of stochastic volatility is included in the model and most importantly, two

[1] http://cfe.cboe.com/education/finaleuromoneyvarpaper.pdf