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# A parametric linear relaxation algorithm for globally solving nonconvex quadratic programming

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ABSTRACT

In this article, we present a parametric linear relaxation algorithm for globally solving the nonconvex quadratic programming (NQP). In this algorithm, a new parametric linearized technique is proposed for generating parametric linear relaxation programming (PLRP) of the NQP, which can be used to determine the lower bound of global minimum value of the NQP. To improve the convergent speed of the proposed algorithm, a pruning operation is employed to compress the investigated region. By subdividing subsequently the initial domain and solving subsequently a series of parametric linear relaxation programming problems over the subdivided domain, the proposed algorithm is convergent to the global minimum of the NQP. Finally, an engineering problem for the design of heat exchanger network and some test examples are used to verify the effectiveness of the proposed algorithm.



## 1. Introduction

In this article, we consider the following nonconvex quadratic programming (NQP) problem:

$$(\mathrm{NQP}):\begin{cases}\min & F_0(x)=x^TQ^0x+(d^0)^Tx,\\ \text{s.t.} & F_i(x)=x^TQ^ix+(d^i)^Tx\leqslant b_i,\quad i=1,2,\ldots,m,\\ & x\in X^0=\{x\in R^n: l^0\leqslant x\leqslant u^0\},\end{cases}$$

where $Q^i=(q^i_{jk})_{n\times n}$ is a symmetric $n\times n$ real matrix, $i=0,1,\ldots,m; d^0,d^i\in R^n, b_i\in R, i=1,2,\ldots,m; l^0=(l^0_1,l^0_2,\ldots,l^0_n)^T, u^0=(u^0_1,u^0_2,\ldots,u^0_n)^T$.

The NQP problem arises naturally in many practical applications including engineering design, optimal control, economic equilibria, Euclidean distance geometry, production planning, combinatorial optimization and so on. Since the NQP problem usually exists many local optimum points which are not global optimum, which puts forward many important theories and computational difficulties, so that it is very necessary to establish an effective algorithm for globally solving the NQP.

In last several decades, many algorithms have been designed for globally solving the NQP and its special forms. Most algorithms for solving the NQP and its special cases are based on relaxations of the problem within a branch-and-bound framework, in which the lower bound is calculated by solving a certain relaxation problem at each node of the branch-and-bound

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search tree. The tightness of the relaxation problem and computational efficiency in computing the lower bound play a very prominent role on affecting the overall effectiveness of the branch-and-bound algorithm. Current relaxation approaches for the NQP are mainly based on various relaxation techniques including linear programming relaxation, convex programming relaxation, and semidefinite programming (SDP) relaxation. For examples, when constraint condition is a single boxed constraint, based on semidefinite programming relaxation within a finite branching scheme, Burer and Vandenbussche [1] proposed a branch-and-bound algorithm for box-constrained nonconvex quadratic programs. When constrained domain is a polyhedral set, this kind of problem is called as linearly constrained quadratic programming (LCQP), Sherali and Tuncbilek [2] developed a reformulation–convexification method for the LCQP in which linear (or convex) programming relaxations are constructed by forming new constraints via suitable products of constraints and variable redefinitions; based on linearization technique, Gao et al. [3] presented a branch and reduce approach for solving the LCQP; based upon SDP relaxation and KKT-branching technique, Burer and Vandenbussche [4] proposed a finite branch-and-bound approach for solving the LCQP. When constrained conditions are ellipsoidal constraints, based on the SDP relaxation, Ye [5] extended a randomized algorithm proposed in [6] for solving the NQP. Fu et al. [7] presented an approximation algorithm for finding the optimal solution of the NQP. Tseng [8] further studied approximation bounds of the SDP relaxation for the NQP. When constraint functions include nonconvex quadratic constraint, based on outer approximation method and linear programming relaxation a branch-and-bound method for solving the NQP was proposed in [9]; based on linear programming relaxation technique, two simplicial branch-and-bound algorithms were presented for solving the NQP in Refs. [10,11], respectively. In Refs. [12,13], a branch-and-cut algorithm based on reformulation–linearization-technique was developed for solving the NQP. In addition, based on the characteristics of quadratic function, Gao [14], Qu [15], Shen et al. [16] and Shen and Liu [17] proposed four different branch and bound approaches for solving the NQP using different linearized techniques, respectively.

In this paper, we shall present a parametric linear relaxation algorithm for globally solving the NQP. The main contributions of this paper are as follows. (1) A new parametric linearized technique is constructed, by utilizing the technique the NQP can be converted into a parametric linear relaxation programming, which can be used to obtain the lower bounds of global minimum value of the NQP, and by successive partition of feasible domain and the solutions of a sequence of parametric linear relaxation programming problems, the proposed algorithm converges to the global minimum of the NQP. (2) By applying the parametric linear relaxations of objective function and constraint functions and the current upper bound of the algorithm, a pruning operation is presented for compressing the investigated region, and which can be used as an accelerating device for improving the running speed of the algorithm. (3) Connecting the bounding operation with partitioning operation and the pruning operation, a parametric linear relaxation algorithm is proposed for globally solve the NQP, and numerical results imply the effectiveness of the algorithm.

The rest of this paper is organized as follows. In Section 2, we first describe a new parametric linearized technique for establishing the parametric linear relaxation programming of the NQP. In Section 3, by connecting the derived parametric linear relaxation programming with the proposed pruning operation in a branch and bound framework, a parametric linear relaxation algorithm is presented for globally solving the NQP. In Section 4, an engineering problem for the design of heat exchanger network and some test examples in recent literatures are used to verify the performance of the algorithm, and numerical results compared with the known methods are presented. At last, some concluding remarks are presented.

## 2. New parametric linearized technique

In this section, to generate the parametric linear relaxation programming of the NQP, a new parametric linearized technique is proposed by underestimating each function $F_i(x)$ with a parametric linear function $F_i^L(x, X, \sigma), i = 0, 1, \ldots, m$. The detailed parametric linearized technique is described as follows:

Let $X = \{x = (x_1, x_2, \ldots, x_n)^T \in R^n : -\infty \leqslant l_j \leqslant x_j \leqslant u_j \leqslant +\infty, j = 1, 2, \ldots, n\}$. Define $\sigma = (\sigma_{jk})_{n\times n} \in R^{n\times n}$ is a symmetric parameter matrix, where $\sigma_{jk} \in \{0, 1\}$. For any $x \in X$, for any $j \in \{1, 2, \ldots, n\}, k \in \{1, 2, \ldots, n\}, j \neq k$, define

$$\begin{aligned}
&x_j(\sigma_{jk}) = l_j + \sigma_{jk}(u_j - l_j), \quad x_k(\sigma_{jk}) = l_k + \sigma_{jk}(u_k - l_k),\\
&x_j(1-\sigma_{jk}) = l_j + (1-\sigma_{jk})(u_j - l_j), \quad x_k(1-\sigma_{jk}) = l_k + (1-\sigma_{jk})(u_k - l_k),\\
&f_{jk}(x) = x_j x_k,\\
&\underline{f}_{jk}(x, X, \sigma_{jk}) = x_j(\sigma_{jk})x_k(\sigma_{jk}) + x_k(\sigma_{jk})(x_j - x_j(\sigma_{jk})) + x_j(\sigma_{jk})(x_k - x_k(\sigma_{jk})),\\
&\overline{f}_{jk}(x, X, \sigma_{jk}) = x_j(\sigma_{jk})x_k(\sigma_{jk}) + x_k(1-\sigma_{jk})(x_j - x_j(\sigma_{jk})) + x_j(1-\sigma_{jk})(x_k - x_k(\sigma_{jk})).
\end{aligned}$$

Obviously, we have

$$x_j(0) = l_j, \quad x_j(1) = u_j, \quad x_k(0) = l_k, \quad x_k(1) = u_k.$$

**Theorem 1.** *For any $x \in X$, for any $j \in \{1, 2, \ldots, n\}, k \in \{1, 2, \ldots, n\}, j \neq k$, consider the function $f_{jk}(x) = x_j x_k$, then we have $\underline{f}_{jk}(x, X, \sigma_{jk}) \leqslant f_{jk}(x) \leqslant \overline{f}_{jk}(x, X, \sigma_{jk})$, and moreover $\underline{f}_{jk}(x(\sigma_{jk}), X, \sigma_{jk}) = \overline{f}_{jk}(x(\sigma_{jk}), X, \sigma_{jk}) = f_{jk}(x(\sigma_{jk}))$.*

**Proof.** The gradient of the function $f_{jk}(x) = x_j x_k$ can be expressed as follows: