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Implementing the complex integral method with the transformed Clenshaw–Curtis quadrature

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ABSTRACT

Gauss–Laguerre quadrature plays an important role in implementing the numerical steepest decent method for computing highly oscillatory integrals. However, it consumes too much time when the analytic region of the integrand is narrow. In this paper, we analyze the convergence rate of the transformed Clenshaw–Curtis quadrature, and show that this method also shares the property that the higher the oscillation, the better the calculation. Moreover, it is efficient to compute highly oscillatory integral with a nearly singularity. Numerical tests are performed to verify our given results.

1. Introduction

In many areas of science and engineering one often encounters the problem of numerical calculation of integrals. In the basic quadrature problem, we are given a function $h \in C([a,b])$ and wish to calculate

$$I[h] = \int_a^b h(t)dt. \tag{1}$$

A standard idea for doing this is substituting the integrand h by its interplant q_n at a certain set of $n+1$ distinct nodes $t_0, \ldots, t_n$ in $[a,b]$, for example, the Clenshaw–Curtis (C–C) and Gauss (G–J) formulas [1–4]. Unfortunately, when the integrand behaves highly oscillatory, the efficiency of these classic methods are often poor [5]. In past several decades, a great many methods have been developed for computation of highly oscillatory integrals [5–11]. Among these techniques, the numerical steepest decent method [6] has proven to be one of the most efficient approaches for the analytical integrand in a large region of the complex plane, which contains the integration interval. The main spirit of this method is transforming the highly oscillatory integrand into a exponentially damped function, then the work remaining is evaluation of the integral like

$$I_r[f] = \int_0^\infty f(t)e^{-rt}dt. \tag{2}$$

Here r is often called the frequency. The asymptotic error is also given in [6]. Generally, the error behaves like $O(r^{-2n-1})$ for n-point Gauss–Laguerre quadrature (G–L), as $r \to \infty$. However, in computing practice, the frequency is often fixed, and little attention has been paid to the convergence when $n \to \infty$. So will this method behaves well in this case? In history, plenty of papers concerns the convergence of Gauss–Laguerre quadrature, for example, [12] for entire functions, [13] for functions of finite regularity, and so on. We show, in Table 1, the computational results of numerical steepest decent method equipped with Gauss–Laguerre quadrature for the oscillatory integral

$$\int_0^\infty \frac{t^2}{25(t+10^{-12})^2} e^{-t+i10t}dt.$$

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It can be seen from this table that the Gauss–Laguerre quadrature converges quite slowly when the integrand behaves nearly singular. Therefore, it is necessary to develop efficient approaches for these integrals, which is also the aim of this paper. In Section 2, we introduce the transformed Clenshaw–Curtis quadrature first. Then we study its convergence rate. Numerical tests in Section 3 verify our given results and show the efficiency of this method for computing Pollaczek integrals.

2. The transformed Clenshaw–Curtis quadrature (T-C–C)

Although the polynomial plays an important role in various approximation problems, an alternative approximation for functions which has a narrow analytic region is the beyond polynomial [14,15]. That is, instead of approximating a function $f(x)$ on $[-1,1]$, we approximate $f(g(s))$ on $[-1,1]$ by a polynomial $p_N(s)$, where $x = g(s)$ denotes a map from $[-1,1]$ to itself. Specially, for the function possessing a singular point $\tilde{z} = \tilde{a} + i\tilde{b}$, we can define a sinh transformation as follows [16].

$$g(s) = \tilde{a} + \tilde{b}\sinh(\tilde{\mu}s - \tilde{\eta}), \tag{3}$$

$$\tilde{\mu} = \frac{1}{2}\left(\operatorname{arcsinh}\left(\frac{1+\tilde{a}}{\tilde{b}}\right) + \operatorname{arcsinh}\left(\frac{1-\tilde{a}}{\tilde{b}}\right)\right), \tag{4}$$

$$\tilde{\eta} = \frac{1}{2}\left(\operatorname{arcsinh}\left(\frac{1+\tilde{a}}{\tilde{b}}\right) - \operatorname{arcsinh}\left(\frac{1-\tilde{a}}{\tilde{b}}\right)\right). \tag{5}$$

This kind of approximation behaves well even when the singular point t comes to the segment $[-1,1]$. Theoretical aspects of the beyond polynomial interplant can be found in [15], and we list results without details of the proof.

Theorem 1. *For a given $\rho > 1$, let functions f and g be analytic in [-1,1] and analytically continuable to the open Bernstein ellipse $\mathcal{E}_\rho$ with $|f(z)| \leqslant M$, then the transformed Clenshaw–Curtis interplant satisfies*

$$\|f - p_N(g^{-1}(x))\|_\infty \leqslant \frac{4M\rho^{-N}}{\rho - 1}. \tag{6}$$

An important application of transformed polynomial approximations is quadrature. In particular, a transformed Clenshaw–Curtis quadrature formula can be obtained by applying Clenshaw–Curtis quadrature to the right term of the following identity

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} f(g(s))g'(s)ds. \tag{7}$$

Now let us consider a special transformed Clenshaw–Curtis quadrature for $I_r[f]$. First, we introduce the definition of a class of functions considered in this paper.

Definition 1. Any function $g(z)$ is said to be in $\mathscr{S}\{[0,\infty)\}$ if it satisfies the following conditions:

- $g(z)$ is analytic for any complex z with $\mathrm{Im}(z) \leqslant 0$;
- $g(z)$ has a singular point $z^* = a + ib$ near the endpoint $z = 0$;
- $g(z)$ is bounded by $M > 0$ on $[0,\infty)$.

Suppose $f(t) \in \mathscr{S}\{[0,\infty)\}$ in (2). Then for any given tolerance error $\epsilon > 0$, letting $A = -\ln(r\epsilon/M)/r$, we obtain

$$\left|\int_A^\infty f(t)e^{-rt}dt\right| \leqslant \int_A^\infty Me^{-rt}dt = \epsilon. \tag{8}$$

Therefore, we can truncate the integral (2) at $t = A$, that is,

$$I_r[f] = \left(\int_0^A + \int_A^\infty\right) f(t)e^{-rt}dt \approx \int_0^A f(t)e^{-rt}dt. \tag{9}$$

Table 1
Relative errors.

$r \setminus N$	100	200	500	1000	2000	5000
10	1.0e−04	1.2e−04	1.3e−04	1.5e−04	1.6e−04	1.8e−04
20	2.0e−04	2.3e−04	2.7e−04	2.9e−04	3.2e−04	3.6e−04
50	5.1e−04	5.7e−04	6.7e−04	7.3e−04	8.0e−04	8.8e−04