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An efficient numerical scheme for a nonlinear integro-differential equations with an integral boundary condition

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ABSTRACT

Nonlinear functional integro-differential equations with an integral boundary condition is appeared in chemical engineering, underground water flow and population dynamics phenomena and other field of physics and mathematical chemistry. So, this paper presents a powerful numerical approach based on an iterative technique and Sinc quadrature to estimate a solution for this equation. Then convergence of this technique is discussed by preparing a theorem which guarantees the applicability of that. Finally, some numerical examples are given to confirm efficiency and accuracy of the numerical scheme.

1. Introduction

Integro-differential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Some of the applications are fluid dynamics, electrodynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, theory of population dynamics, compartmental systems, nuclear reactors. For details, see [1,2] and the references therein.

In this article, we consider the nonlinear functional integro-differential equation with an integral boundary condition

$$\begin{cases} v'(x) = f\left(x, \int_0^x H(x,z)\Psi(v(z))dz, \int_0^1 K(x,v(z))dz\right), & 0 \leqslant x \leqslant 1, \\ v(0) = \lambda v(1) + \int_0^1 D(z)v(z)dz, & \lambda \in \mathbb{R}, \end{cases} \tag{1}$$

where $H(\cdot,\cdot)$, $\Psi(\cdot)$, $K(\cdot,\cdot)$ are known functions, and $v(\cdot)$ is unknown which should be determined. It is well known that nonlocal conditions came up when the values of function on the boundary were connected to the values inside the domain. So, integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, underground water flow, population dynamics, and so forth. For a detailed description of the nonlocal and integral boundary conditions, we refer the reader to [3–5].

There were extensive literatures on non-local problems and the boundary value problems (BVPs) involving integral boundary conditions. Some methods were applied to solve integro-differential equation with an integral boundary condition such as functional method, energy method, Galerkin method and discretization method. In [6], the authors used the method of lower and upper solutions combined with monotone iterative techniques successfully for problem (1). For BVPs with

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integral boundary conditions and comments on their importance, we refer the reader to the papers [7–9] and the references therein. However, the literature of numerical analysis contains little on the solution of (1).

It is easy to convert problem 1 to functional integral equation

$$v(x) = \lambda v(1) + \int_0^1 D(z)v(z)dz + \int_0^x f\left(t, \int_0^t H(t,z)\Psi(v(z))dz, \int_0^1 K(t,v(z))dz\right)dt, \tag{2}$$

where $0 \leqslant x \leqslant 1$ and in this paper we try to discuss on solution of problem (1) in point of equivalent functional integral equation (2). Also, investigation on existence theorems for some nonlinear functional integral equations has been presented in other references such as [10,11].

Functional integral equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Numerical methods for solving integro-differential and integral equations and investigating on existence and uniqueness of solution of some general models have been studied by many authors so far [10–20]. Some of them usually use techniques based on a projection in terms of some basis functions or use some quadrature formulas, and the convergence rate of these methods are usually of polynomial order with respect to N, where N represents the number of terms of the expansion or the number of points of the quadrature formula. These methods often transform an integral or an integro-differential equation into a linear or nonlinear system of algebraic equations that can be solved by direct or iterative methods. On the other hand, in [21] it is shown that if we use the Sinc method the convergence rate is $O(exp(-c\sqrt{N}))$ with some $c > 0$. Although this convergence rate is much faster than that of polynomial order.

So, in the present paper, we apply the Sinc-quadrature formula and an iterative method to estimate a numerical solution for Eq. (2). Our method does not consist of reducing the solution of Eq. (2) to a set of algebraic equations by expanding $v(x)$ as Sinc functions with unknown coefficients, so this scheme has less computations and exponential accuracy. Also, in the following, a theorem is prepared to guarantee the convergence of numerical scheme.

This paper is organized as follows. In Section 2, we state basic theorems and properties of the Sinc function which are referred in the subsequent sections. In Section 3, Sinc-quadrature method is derived by means of the Sinc approximation and then an iterative numerical method is prepared, and in continue convergence analysis of numerical technique applied for Eq. (2). In the last section, numerical examples are given to show accuracy and applicability of the numerical technique.

2. Sinc function properties

In this section, we introduce the cardinal function and some of its properties. For this result $sinc(x)$ definition is followed by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Now, for $h > 0$ and integer k, we define kth Sinc function with step size h by

$$S(k,h)(x) = \frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h}.$$

2.1. Sinc approximation on real line

The Sinc approximation on the entire interval $(-\infty, \infty)$ is defined as

$$f(x) \approx \sum_{k=-N}^{N} f(kh)S(k,h)(x).$$

Now, the following Definition and Theorem will guarantee the approximation authority of Sinc functions on the real line.

Definition 1. Let $H^1(D_d)$ denote the family of all functions analytic in D_d defined by

$$D_d = \{z \in \mathcal{C} : |Im(z)| < d\}$$

such that for $0 < \epsilon < 1$, $D_d(\epsilon)$ is defined by

$$D_d(\epsilon) = \left\{z \in \mathcal{C} : |Im(z)| < d(1-\epsilon), |Re(z)| < \frac{1}{\epsilon}\right\},$$

then $N(f, D_d) < \infty$ with

$$N(f, D_d) = \lim_{\epsilon \to 0}\left(\int_{\partial D_d} |f(z)||dz|\right).$$