Applied Mathematics and Computation 248 (2014) 4–7

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On conditioning of saddle-point matrices with Lagrangian augmentation ☆

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ARTICLE INFO

Keywords:
Linear system
Matrix splitting
Series expansion
Truncated inverse
Preconditioning

ABSTRACT

For saddle-point matrix and its Lagrangian augmentation, we derive bounds on eigenvalues and on Euclidean condition numbers, and discuss the asymptotic behavior of the Euclidean condition numbers with respect to the parameter in the augmentation.

1. Introduction

Let

$$A \equiv \begin{pmatrix} B & E \\ E^* & 0 \end{pmatrix} \tag{1.1}$$

be a saddle-point matrix, with $B \in \mathbb{C}^{p\times p}$ being nonsingular, Hermitian and indefinite, and $E \in \mathbb{C}^{p\times q}$ being of full column-rank. Note that the matrix A is Hermitian and indefinite [2,5]. Let

$$A_\gamma \equiv \begin{pmatrix} B_\gamma & E \\ E^* & 0 \end{pmatrix} \tag{1.2}$$

be the corresponding saddle-point matrix with Lagrangian augmentation, where

$$B_\gamma = B + \gamma E^* W^{-1} E, \tag{1.3}$$

with γ being a positive constant and $W \in \mathbb{R}^{q\times q}$ being a Hermitian positive definite matrix [3]. Here and in the sequel, we use $(\cdot)^*$ to denote the conjugate transpose of either a vector or a matrix, and represent by $n = p + q$.

In this paper, for saddle-point matrix A and the augmented matrix A_γ, we derive bounds on eigenvalues and bound on Euclidean condition number of the augmented matrix A_γ in terms of those of the matrix A, and discuss the asymptotic behavior of the Euclidean condition number $\kappa(A_\gamma)$ with respect to the parameter γ. Alternatively, we also derive bounds on eigenvalues and bound on Euclidean condition number of the matrix A in terms of those of the augmented matrix A_γ. These estimates further improve those given in [4].

2. Conditioning analysis of A_γ

In [4], based on the matrix identity

☆ Supported by the National Natural Science Foundation (Nos. 11401032; 11371075; 61473325), P.R. China.
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http://dx.doi.org/10.1016/j.amc.2014.09.087
0096-3003/

$$A_\gamma^{-1} = A^{-1} - \mathrm{Diag}(0, \gamma W^{-1}) \tag{2.1}$$

the authors presented the following upper bound on the condition number of the matrix A_γ:

$$\kappa(A_\gamma) \leqslant \kappa(A) + \gamma(\|E\|^2 + 1)\|W^{-1}\|\|A^{-1}\| + \gamma^2\|W^{-1}\|^2\|E\|^2.$$

As a result, when $W = I$, the identity matrix, it holds that

$$\frac{\kappa(A_\gamma)}{\gamma^2} \longrightarrow \|E\|^2$$

as $\gamma \to \infty$.

Also, based on (2.1), we can demonstrate the following bounding the eigenvalues of the augmented matrix A_γ, with the notations that $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ represent the smallest and the largest eigenvalues of a Hermitian matrix M, and $\mathrm{sp}(M)$ indicates the set of all of its eigenvalues.

Theorem 2.1. *Let* $\mathrm{sp}(A) \subseteq [-\Theta, -\theta] \cup [\delta, \Delta]$, *with* θ, Θ *and* δ, Δ *being positive reals. Then* $\mathrm{sp}(A_\gamma) \subseteq [-\Theta_\gamma, -\theta_\gamma] \cup [\delta_\gamma, \Delta_\gamma]$, *where* $\Theta_\gamma = \Theta, \delta_\gamma = \delta$, *and*

$$\theta_\gamma = \frac{\theta}{1 + \gamma\theta\|W^{-1}\|}, \qquad \Delta_\gamma = \Delta + \gamma\|W^{-1/2}E\|^2.$$

Proof. From

$$A_\gamma = A + \mathrm{Diag}(\gamma E^* W^{-1} E, 0)$$

we can straightforwardly obtain the estimates

$$\lambda_{\max}(A_\gamma) \leqslant \Delta + \gamma\|W^{-1/2}E\|^2 \quad \text{and} \quad \lambda_{\min}(A_\gamma) \geqslant -\Theta,$$

which imply that

$$\Delta_\gamma = \Delta + \gamma\|W^{-1/2}E\|^2 \quad \text{and} \quad \Theta_\gamma = \Theta.$$

And from (2.1) and $\mathrm{sp}(A^{-1}) \subseteq \left[-\frac{1}{\theta}, \frac{1}{\delta}\right]$ we can derive the bounds

$$\lambda_{\max}(A_\gamma^{-1}) \leqslant \lambda_{\max}(A^{-1}) \leqslant \frac{1}{\delta}$$

and

$$\lambda_{\min}(A_\gamma^{-1}) \geqslant \lambda_{\min}(A^{-1}) - \gamma\|W^{-1}\| \geqslant -\frac{1}{\theta} - \gamma\|W^{-1}\|.$$

It then follows from Lemma 2.2 in [1] that

$$\frac{1}{\delta_\gamma} = \frac{1}{\delta} \quad \text{or} \quad \delta_\gamma = \delta,$$

and

$$-\frac{1}{\theta_\gamma} = -\frac{1}{\theta} - \gamma\|W^{-1}\| \quad \text{or} \quad \theta_\gamma = \frac{\theta}{1 + \gamma\theta\|W^{-1}\|}. \quad \square$$

Theorem 2.1 immediately results in an estimate about the Euclidean condition number $\kappa(A_\gamma)$ of the augmented matrix A_γ.

Theorem 2.2. *Let* $\mathrm{sp}(A) \subseteq [-\Theta, -\theta] \cup [\delta, \Delta]$, *with* θ, Θ *and* δ, Δ *being positive reals. Then*

$$\begin{aligned}\kappa(A_\gamma) &\leqslant \frac{\left(\max\{\Delta, \Theta\} + \gamma\|W^{-1/2}E\|^2\right)\left(1 + \gamma\theta\|W^{-1}\|\right)}{\min\{\delta, \theta\}} \\ &= \kappa(A) + \frac{\left(\|W^{-1/2}E\|^2 + \theta\max\{\Delta, \Theta\}\|W^{-1}\|\right)\gamma + \theta\|W^{-1}\|\|W^{-1/2}E\|^2\gamma^2}{\min\{\delta, \theta\}}.\end{aligned}$$