Applied Mathematics and Computation 248 (2014) 47–54

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the metric dimension of circulant and Harary graphs

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ARTICLE INFO

Keywords:
Metric basis
Metric dimension
Circulant graphs
Harary graphs

ABSTRACT

A metric generator is a set W of vertices of a graph $G(V,E)$ such that for every pair of vertices u, v of G, there exists a vertex $w \in W$ with the condition that the length of a shortest path from u to w is different from the length of a shortest path from v to w. In this case the vertex w is said to resolve or distinguish the vertices u and v. The minimum cardinality of a metric generator for G is called the metric dimension. The metric dimension problem is to find a minimum metric generator in a graph G. In this paper, we make a significant advance on the metric dimension problem for circulant graphs $C(n, \pm\{1,2,\ldots,j\})$, $1 \leqslant j \leqslant \lfloor n/2 \rfloor$, $n \geqslant 3$, and for Harary graphs.

1. Introduction

Let $G(V,E)$ be a simple connected and undirected graph. For $x, y \in V$ let $d(x,y)$ denote the distance between x and y. A vertex $v \in V$ is said to resolve or distinguish two vertices x and y if $d(v,x) \neq d(v,y)$. A set $W \subseteq V$ is said to be a metric generator for G if any pair of vertices of G is distinguished by some element of W. A minimum metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $dim(G)$. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the k-vector (ordered k-tuple) $code(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$ as the *code* (or *representation*) of v with respect to W. Thus we can have another equivalent definition. The set W is called a *metric generator* if distinct vertices of G have distinct codes with respect to W. The metric dimension problem has been studied in different papers including [7,11,28,31], where it is also referred to as the *location number*. Note that metric basis, *locating set* and *resolving set* are different names used by different authors to describe the same concept. In this paper we use the terms metric basis and metric dimension.

The problem of finding the metric dimension of a graph was studied by Harary and Melter [11]. Slater described the usefulness of this idea in long range aids to navigation [28]. Melter and Tomescu [23] studied the metric dimension problem for grid graphs. The metric dimension problem has been studied also for trees, multi-dimensional grids [18] and torus networks [19]. The concept of metric dimension has some applications in chemistry for representing chemical compounds [17] as well as in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures. Other applications include navigation of robots in networks and other areas [5,12]. Khuller et al. [18] describe the application of this problem in the field of computer science and robotics. It is interesting to learn [11] that a graph has metric dimension 1 if and only if it is a path. If G has n vertices then it is clear that $1 \leqslant dim(G) \leqslant n-1$. Also $dim(K_n) = n-1$, $dim(C_n) = 2$, and $dim(K_{m,n}) = m+n-2$, where K_n, C_n, and $K_{m,n}$ are the complete graph, the cycle, and the complete bipartite graph, respec-

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http://dx.doi.org/10.1016/j.amc.2014.09.045
0096-3003/

tively [11]. Garey and Johnson [10] claimed that the problem of metric dimension is NP-complete for general graphs by a reduction from 3-dimensional matching. Manuel et al. [20] have proved that the metric dimension problem is NP-complete for bipartite graphs by a reduction from 3-SAT, thus narrowing down the gap between the polynomial classes and NP-complete classes of the metric dimension problem. The problem has been studied for Beneś network, honeycomb network and certain binary tree derived architectures [20–22]. The metric dimension of Cartesian product of graphs was studied in [6]. In 2009, Saputro et al. studied the metric dimension of a complete n-partite graph and its Cartesian product with a path [26]. The metric dimension of the lexicographic product [27] and corona product of graphs was studied in detail in [30]. Rodríguez-Velázquez et al. obtained closed formulae and tight bounds for the metric dimension of strong product graphs [25]. Iswadi et al. [14,15] discussed the metric dimension problem of the amalgamation of cycles. A variant of location number, locating-chromatic number of the amalgamation of stars and firecracker graphs was studied in [1,2], respectively. Recently Estrada-Moreno et al. [8] gave a generalization of the concept of metric basis, the notion of k-metric basis in graphs, and tight bounds and closed formulae for the k-metric dimension of connected corona graphs [9] were also obtained. In this paper, we obtain the metric dimension of circulant graphs and Harary graphs.

2. Circulant graphs

A circulant graph is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [29]. Circulant graphs have been used for decades in the design of computer and telecommunication networks; their popularity is due to their optimal fault-tolerance and routing capabilities [4]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [3]. Every circulant graph is a vertex transitive graph as well as a Cayley graph. Most of the earlier research concentrated on using circulant graphs to build interconnection networks for distributed and parallel systems [3,4].

More formally, a circulant graph, denoted by $C(n, \pm\{1,2,\ldots,j\}), 1 \leqslant j \leqslant \lfloor n/2 \rfloor, n \geqslant 3$, is defined as a graph with vertex set $V = \{0,1,\ldots,n-1\}$ and edge set $E = \{(i,j) : |j-i| \equiv s(mod\ n), s \in \{1,2,\ldots,j\}\}$. It is clear that $C(n,\pm 1)$ is an undirected cycle C_n and $C(n, \pm\{1,2,\ldots,\lfloor n/2 \rfloor\})$ is the complete graph K_n. The cycle C_n is a subgraph of $C(n, \pm\{1,2,\ldots,j\})$, for every $j, 1 \leqslant j \leqslant \lfloor n/2 \rfloor$, and is sometimes referred to as the outer cycle.

Let F be a family of connected graphs $G_n : F = (G_n)_{n \geqslant 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G_n) \leqslant C$, for every $n \geqslant 1$, then we shall say that F has bounded metric dimension [13].

The metric dimension of a class of circulant graphs $C(n, \pm\{1,2\})$ has been determined in [24] as follows.

Theorem 2.1 [24]. *Let $G = C(n, \pm\{1,2\})$. Then,*

$$dim(G)\begin{cases} = & 3 \quad when\ n \equiv 0,2,3(mod\ 4) \\ \leqslant & 4 \quad when\ n \equiv 1(mod\ 4) \end{cases}.$$

Later Imran et al. [13] extended this study to the class of circulant graphs $C(n, \pm\{1,2,3\})$ and proved the following theorem.

Theorem 2.2 [13]. *Let $G = C(n, \pm\{1,2,3\})$. Then*

$$dim(G)\begin{cases} = & 4 \quad when\ n \equiv 2,3,4,5(mod\ 6) \\ \leqslant & 5 \quad when\ n \equiv 0(mod\ 6) \\ \leqslant & 6 \quad when\ n \equiv 1(mod\ 6) \end{cases}.$$

In this article we discuss the metric dimension of $G = C(n, \pm\{1,2,\ldots,j\})$, where $j < \lfloor n/2 \rfloor$, and show that G has bounded metric dimension. The integer $j < \lfloor n/2 \rfloor$ is arbitrary but is fixed throughout the paper.

2.1. Lower bound for the metric dimension of circulant graphs

To establish a lower bound and an upper bound of the metric dimension we make use of the following lemma, where all integers are taken modulo n.

Lemma 2.3. *Let $G = C(n, \pm\{1,2,\ldots,j\}), 1 < j \leqslant \lfloor n/2 \rfloor, n \geqslant 3$, be a circulant graph of diameter λ. Let $V = \{0,1,2,\ldots,n-1\}$. Then two vertices $i, i+1$ are resolved by any of the vertices $n+i+1+kj$ or $n+i-kj$, where $1 \leqslant k \leqslant \lambda$.*

Proof. Let w be a vertex such that $w \neq n+i+1+kj,\ 1 \leqslant k \leqslant \lambda$. Both i and $i+1$ are at equal distance from w, *i.e.*, $d(i,w) = d(i+1,w) = \lceil \frac{|i-w|}{j} \rceil$. From the vertex $n+i+1+kj$, vertices i and $i+1$ are at distance $k+1$ and k, respectively. Similar arguments hold also for $n+i-kj$. □

The following definitions will be useful.