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The singular function boundary integral method for an elastic plane stress wedge beam problem with a point boundary singularity

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ABSTRACT

The singular function boundary integral method (SFBIM) is applied for the numerical solution of a 2-D Laplace model problem of a perfectly elastic wedge beam under plane stress conditions. The beam has a point boundary singularity, it includes a curved boundary part and is subjected to non-trivial distributed external loading. The implemented solution method converges for this special model problem extremely fast. The numerical estimates attained for the leading singular coefficients of the local asymptotic expansion and the stress and strain fields are highly accurate, as verified by comparison with the available analytical solution.

1. Introduction

In the area of linear elasticity, there exist many problems described by the Laplace equation in either two or three dimensions. When boundary singularities are present, caused either by an abrupt change in the boundary conditions or by a re-entrant corner, one needs to compute the singular coefficients of the solution expansion in the neighborhood of the singularity, which is intended to represent the Airy stress function Φ. In the case of a plane problem in polar coordinates, this expansion is of the form:

$$\Phi(r,\theta) = \sum_{j=1}^{\infty} \beta_j r^{\lambda_j} f_j(\theta), \tag{1}$$

where the polar coordinates (r, θ) are centered at the singular point. The eigenvalues λ_j and the eigenfunctions f_j of the problem are determined by the boundary conditions along the boundary parts causing the singularity. The values of the unknown singular coefficients β_j, determined by the boundary conditions at the rest of the boundary, are of significant importance in many applications. In fracture mechanics these coefficients are also known as generalized stress intensity factors [1].

In the past few decades, many special numerical methods have been proposed for the solution of elliptic boundary value problems with boundary singularities, in order to overcome difficulties related to the lack of adequate accuracy and to poor convergence in the neighborhood of singularity points. Remedies used were special mesh refinement schemes, multigrid methods, singular elements, *p*/*hp* finite elements and many other techniques (see, e.g. [2–5]). An extensive survey of the treatment of singularities in elliptic boundary value problems is provided in the review article by Li and Lu [6].

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Certain techniques incorporate the form of the local asymptotic expansion [7]. For example, Georgiou and co-workers [8,9] developed a singular finite element method, in which special elements are employed. With their method, the radial form of the local singularity expansion is employed, in the neighborhood of the singularity, in order to resolve the convergence difficulties and improve the accuracy of the global solution. In the Finite Element Method (FEM), the singular coefficients are calculated by post-processing of the numerical solution. Especially with high-order *p* and *hp* FEM versions, fast convergence is achieved by: (i) increasing the degree of the piecewise polynomials (in the case of the *p* version) and (ii) by decreasing the characteristic size *h* of the elements and increasing *p* (in the case of the *hp* version). Such methods proved very successful in solving elliptic boundary value problems with a boundary singularity [10,11]. Also, some interesting post-processing procedures have been proposed for the calculation of the singular coefficients from the finite element solution [3,4].

In the past two decades, Georgiou and co-workers [12–18] developed and tested the Singular Function Boundary Integral Method (SFBIM), in which the unknown singular coefficients are calculated directly, thus giving directly the approximation of the Airy stress function in Laplacian and biharmonic problems of plane elasticity, or the stream-function in biharmonic problems of fluid mechanics. Recently, an extension of the method has been made for 3-D elliptic problems of elasticity [17]. With the SFBIM the solution is approximated by the leading terms of expansion (1) and the Dirichlet conditions on boundary parts away from the singularity are enforced by means of Lagrange multipliers. Numerical studies, as well as theoretical analyses, demonstrated that the SFBIM exhibits exponential convergence with the number of singular functions and can achieve high solution accuracy [12–18].

In the present work, the SFBIM is applied for the solution of a 2-D Laplace model problem of a perfectly elastic wedge beam under plane stress conditions. The beam has a curved boundary part, the center of curvature of which does not coincide with the point of boundary singularity. The distributed external loads applied to the beam impose non-standard boundary conditions, which would be difficult to handle in the framework of the classical FEM or other approaches. The analytical expression of the Airy stress function for this problem is known, which allows for objective accuracy assessment of the implemented numerical method. Hence, our objectives are: (a) to efficiently solve this special model problem without transforming it into an equivalent biharmonic problem (with the angle of the wedge being less than $3\pi/4$, it is impossible to find a real function for the local solution expansion); and (b) to study the convergence and the accuracy of the SFBIM. The model problem is described in Section 2. In Section 3, the formulation of the SFBIM is presented. Section 4 reports and discusses the numerical results obtained and demonstrates the fast convergence of the SFBIM. Finally, the conclusions are summarized in Section 5.

2. A model problem with a point boundary singularity

We consider the plane stress problem of the 2-D unit-thickness wedge beam schematically illustrated in Fig. 1. The beam has two free straight boundary parts, which are subjected to distributed normal and shear loads, as well as a supported curved boundary part. More specifically, distributed normal pressures $p(r)$ act vertically and horizontally along the boundary segment OA of length L, while a distributed shear load $\tau(r)$ acts along the inclined boundary segment OB. External loading consists of these distributed loads only; there are no concentrated external forces on the structure. Moreover, the self-weight of the beam is ignored. The third boundary segment AB, which is fixed-supported and curved, is part of the circumference of a circle with center at point K and radius R; the y-coordinate of K is equal to $H/2$, which is half the length of chord AB. The singularity of this problem is at the free tip O of the wedge, which lies at the intersection of the two straight loaded boundary segments OA and OB. The center K of the circle defining the curved boundary part AB is far away from the singular point O.

The physical boundary conditions of this model problem are as follows:

$$\left.\begin{array}{ll}\sigma_{rr} = -p(r) = -3r - 0.01r^7, \quad \sigma_{\theta\theta} = p(r) = -\sigma_{rr}, \quad \sigma_{r\theta} = 0 & \text{on OA} \\ \sigma_{rr} = \sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = \tau(r) = 3r - 0.01r^7 & \text{on OB} \\ u_r = u_\theta = 0 & \text{on AB}\end{array}\right\}, \qquad (2)$$

where the stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ and the displacements u_r and u_θ can easily be deduced from $\Phi(r,\theta)$ expressed in polar coordinates. Note that all expressions are dimensional; distributed loads are expressed in kN/m and lengths in m (L = 3 m, R = 10 m). Fig. 2 gives a graphical presentation of the distributed loads $p(r)$ and $\tau(r)$ acting along boundaries OA and OB, respectively, in order to provide a view of the non-trivial loading conditions of the structure analyzed. The material

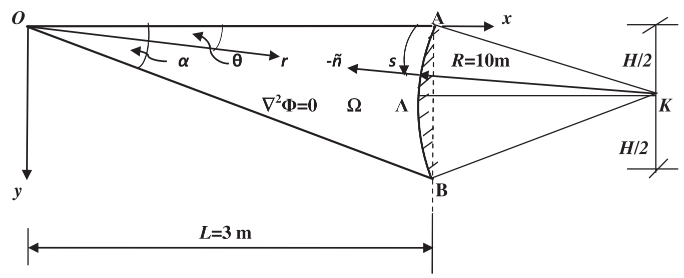

Fig. 1. Schematic illustration of the 2-D wedge beam.