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Strong convergence of asymptotically pseudocontractive semigroup by viscosity iteration

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ABSTRACT

In this paper, we study the strong convergence of viscosity iteration and modified viscosity iteration process for strongly continuous semigroup of uniformly Lipschitzian asymptotically pseudocontractive mappings.

1. Introduction

Analytical and numerical construction of fixed points of nonexpansive mappings, and of common fixed points of nonexpansive semigroups, became in recent years important topics in Optimization Theory; please, see [1] Agarwal et al., Takahashi [2,3] Aoyama et al., Li et al. [4], Chang [5]. That is why they found various utilizations in a large number of applied areas. We have in mind image recovery and signal processing; Byrne [6], Podilchuk and Mammone [7], Sezan and Stark [8], Youla [9,10]. The most straightforward way to study nonexpansive mappings is to use contraction mapping to approximate fixed point of nonexpansive mapping; Browder [11], Browder and Petryshyn [12], Deimling [13], Reich [14,15], Shou [16], Suzuki [17], Xu [18].

Viscosity method provides an efficient approach to a large number of problems coming from different branches of Mathematical Analysis. A major feature of these methods is to provide as a limit of the solution of the approximate problems, a particular solution of the original problem, called a *viscosity solution*. It has been successfully applied to various problems coming from calculus of variations, minimal surface problems, plasticity theory and phase transition; Kohn and Sternberg [19], Ladyzenskaya and Uralceva [20], Lions [21]. Various applications of the viscosity methods can be found in optimal control theory, singular perturbations, minimal cost problem; Attouch [22], Lions [23,24], and in stochastic control theory; Fleming and McEneaney [25]. First abstract formulation of the properties of the viscosity approximation have been given by Tykhonov [26] in 1963 when studying ill-posed problems; see Dontchev and Zolezzi [27] for details. The concept of viscosity solution for Hamilton–Jacobi equations, which plays a crucial role in control theory, game theory and partial differential equations has been introduced by Crandall and Lions [28]; also, see Cho and Kang [29].

In 2000, Moudafi [30] introduced a viscosity approximation method to compute fixed points of nonexpansive mappings. Xu [31] studied further the viscosity approximation method for nonexpansive mapping in uniformly smooth Banach spaces, while Song and Xu [33] studied the convergence of their implicit viscosity iterative scheme for nonexpansive semigroup. Song and Chen [34] proposed implicit viscosity iterative scheme for a fixed Lipschitz strongly pseudocontractive mapping and a continuous pseudocontractive mapping.

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In this paper, we introduce our iteration schemes for a strongly continuous asymptotically pseudocontractive semigroup. The results presented in the paper extend, improve and generalize the corresponding results of Li et al. [4], Browder [11], Reich [14], Moudafi [30], Xu [31,32], Song and Chen [34], Xu and Ori [35], Chidume [36], Li and Gu [37] and others.

2. Preliminaries

Let K be a nonempty subset of a real Banach space E and let $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|;\ \|x\| = \|f\|\}, \quad \forall x \in E, \tag{2.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single valued. In the sequel, we shall denote the single valued normalized duality mapping by j.

We recall that a mapping $T : K \to K$ is called

(i) contraction, if there exists a constant $\beta \in (0,1)$ such that

$$\|Tx - Ty\| \leqslant \beta\|x - y\|, \quad \forall x, y \in K;$$

(ii) nonexpansive, if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in K;$$

(iii) pseudocontractive [12], if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leqslant \|x - y\|^2, \quad \forall x, y \in K; \tag{2.2}$$

(iv) strongly pseudocontractive, if there exists a constant $\beta \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leqslant \beta\|x - y\|^2, \quad \forall x, y \in K; \tag{2.3}$$

(v) asymptotically pseudocontractive [16], if there exists a sequence $\{k_n\} \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that

$$\left\langle T^n x - T^n y, j(x-y)\right\rangle \leqslant k_n\|x - y\|^2, \quad \forall x, y \in K, \quad \forall\, n \geqslant 1; \tag{2.4}$$

(vi) uniformly L-Lipschitzian, if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \leqslant L\|x - y\|, \quad \forall x, y \in K, \quad \forall\, n \geqslant 1.$$

Let K be a closed convex subset of a Banach space E, and $\mathbb{R}^+$ denote the set of nonnegative real numbers. A family $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ of asymptotically pseudocontractive mappings from K into K is called strongly continuous semigroup of asymptotically pseudocontractive mappings, Chidume [36], if the following conditions are satisfied:

(i) $T(0)x = x$, for all $x \in K$;
(ii) $T(s+t)x = T(s)T(t)x$, for all $x \in K$ and all $s, t \in \mathbb{R}^+$;
(iii) for each $x \in K$, the mapping $t \mapsto T(t)x$ is continuous for $t \in \mathbb{R}^+$;
(iv) there exists $\{k_n\} \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that

$$\left\langle (T(t_n))^n x - (T(t_n))^n y, j(x-y)\right\rangle \leqslant k_n\|x - y\|^2, \quad \forall t_n \geqslant 0,\ \forall x, y \in K. \tag{2.5}$$

$\mathcal{T}$ is said to be strongly continuous semi-group of

(i) uniformly L-Lipschitzian if there exists $L > 0$ such that

$$\left\|(T(t_n))^n x - (T(t_n))^n y\right\| \leqslant L\|x - y\|, \quad \forall x, y \in K,\ \forall n \geqslant 1,\ \forall t_n \geqslant 0;$$

(ii) uniformly asymptotically regular if

$$\lim_{n\to\infty}\left\|(T(t))^{n+1}x - (T(t))^n x\right\| \to 0, \quad \forall\, t \geqslant 0,\ \forall x \in K.$$

$\mathcal{T}$ is said to have a fixed point if there exists $x_0 \in K$ such that $T(t)x_0 = x_0$ for all $t \geqslant 0$. We denote the set of fixed points of $\mathcal{T}$ by $F(\mathcal{T}) = \bigcap_{t\in\mathbb{R}^+} F(T(t))$.

For given $u \in K$ and each $t \in (0,1)$, define a contraction $T_t : K \to K$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in K.$$