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A new tool to study real dynamics: The convergence plane

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ABSTRACT

In this paper, the author presents a graphical tool that allows to study the real dynamics of iterative methods whose iterations depends on one parameter in an easy and compact way. This tool gives the information as previous tools such as Feigenbaum diagrams and Lyapunov exponents for every initial point. The convergence plane can be used, inter alia, to find the elements of a family that have good convergence properties, to see how the basins of attraction changes along the elements of the family, to study two-point methods such as Secant method or even to study two-parameter families of iterative methods. To show the applicability of the tool an example of the dynamics of the Damped Newton's method applied to a cubic polynomial is presented in this paper.

1. Introduction and motivation

The main aim of the paper is to present a new tool which can make easier the study of the real dynamics of families of iterative methods which depends on a certain parameter or even the study of an iterative method applied to a uniparametric family of polynomials. This tool can be modified in order to extend, amongst other ones, to methods which need two approximations as for example, secant-type methods, modified Newton's method, etc.

The dynamics of iterative methods used for solving nonlinear equations in complex plane have been studied recently by many authors [3,4,9–12,14,15,17,18]. There exists a belief that real dynamics is included in the complex dynamics but this is not true at all. For example in the real dynamics one can prove the monotone convergence which does not exist in the complex plane, there exist also asymptotes in the real dynamics but in the complex one that concept has no sense or the point $z = \infty$ in the complex plane can be studied as another point but in the real line it is not possible. As a consequence real dynamics are not contained in the complex dynamics and both must be studied separately. Taking into account that distinction, many authors [1,11,19,15] have begun to study it since few years ago.

If one focuses the attention on families of iterative methods studied in the complex plane, parameter spaces have given rise to methods whose dynamics are not well-known. These parameter spaces consist on studying the orbits of the free critical points associating each point of the plane with a complex value of the parameter. Several authors [9–11,15] have studied really interesting dynamical and parameter planes in which they have found some anomalies such as convergence to n-cycles, convergence to ∞, or even chaotical behavior. In the real line, there exists tools such as Feigenbaum diagrams or Lyapunov exponents that allow us to study what happens with a concrete point, but it is really hard to study each point in a separate way. This is the main motivation of the author to present the new tool which will be called "The Convergence Plane".

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The Convergence Plane is obtained by associating each point of the plane with a value of the starting point and a value of the parameter. That is, the tool is based on taking the vertical axis as the value of the parameter and the horizontal axis as the starting point, so every point in the plane represents an initial estimation and a member of the family. If one draws a straight horizontal line in a concrete value of the parameter, the dynamical behavior, for that value, for every starting point is on that line. On the other hand, if the straight line is vertical, the dynamical behavior for that starting point and every value of the parameter is on that line, this is the information that gives Feigenbaum diagrams or Lyapunov exponents, so the information given by both tools is included.

The rest of the paper is organized as follows: in Section 2 some basic concepts are shown, in Section 3 the Algorithm of *The Convergence Plane* is shown and in Section 4 an example of The Convergence Plane associated to the Damped Newton's method applied to the polynomial $f_-(x) = x^3 - x$ is provided in order to validate the tool. Finally, the main conclusions are shown in Section 5.

2. Basic notions of real dynamics

One of the most frequently problems in Sciences and more specifically in Mathematics is solving a nonlinear equation

$$f(x) = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$. The solutions of this equations can rarely be found in a direct way. That is why most of the methods for solving these equations are iterative.

In this work, we are concerned with the problem of studying one-parametric families of iterative methods

$$x_{n+1} = S_\lambda(x_n), \tag{2.1}$$

where λ is a real parameter. There exists several families of iterative methods which depend on one parameter as for example the Chebyshev-Halley family (see [10]), the fourth-order family of Jarratt iterative method introduced by Amat et al. in [3], the Damped Newton method [8,15], some of the iterative methods given by Argyros in [5–7] and others [2,16]. The dynamical properties related to a family of iterative methods applied to polynomials give important information about its stability and reliability.

In this section, some dynamical concepts of real dynamics that will be used in this work are shown. As the vast majority of iterative methods applied to polynomials gives a rational function, the focus will be centered in them. Given a rational function $S : \mathbb{R} \to \mathbb{R}$, the *orbit of a point* $x_0 \in \mathbb{R}$ is defined as

$$\left\{x_0, S(x_0), S^2(x_0), \ldots, S^n(x_0), \ldots\right\}.$$

A point $x_0 \in \mathcal{R}$, is called a *fixed point* of $S(z)$ if it verifies that $S(z) = z$. There exist different types of fixed points depending on its associated multiplier $|S'(x_0)|$. Taking the associated multiplier into account a fixed point x_0 is called:

- *superattractor* if $|S'(x_0)| = 0$,
- *attractor* if $|S'(x_0)| < 1$,
- *repulsor* if $|S'(x_0)| > 1$
- and *parabolic* if $|S'(x_0)| = 1$.

The fixed points that do not correspond to the roots of the polynomial $f(x)$ are called *strange fixed points*. On the other hand, a *critical point* x_0 is a point which satisfies the condition $S'(x_0) = 0$.

The basin of attraction of an attractor α is defined as

$$\mathcal{A}(\alpha) = \{x_0 \in \mathbb{R} : \ S^n(x_0) \to \alpha, \ n \to \infty\}.$$

Moreover, a point x_0 is called a *periodic point* of period $p > 1$ if it is a point such that $S^p(x_0) = x_0$ but $S^k(x_0) \neq x_0$, for each $k < p$. Moreover, a point x_0 is called *pre-periodic* if it is not periodic but there exists a $k > 0$ such that $S^k(x_0)$ is periodic.

On the other hand, the orbit of a periodic point of period n is called n-cycle. A n-cycle could be.

- *superattractive* if $|\mu| = 0$,
- *attractive* if $|\mu| < 1$,
- *repulsive* if $|\mu| > 1$,
- and *parabolic* if $|\mu| = 1$,

where $\mu = |(S^n)'(x_0)| = \cdots = |(S^n)'(x_{n-1})| = |S'(x_0)||S'(x_1)||S'(x_2)|\cdots|S'(x_{n-1})|$.

The *Fatou set* of the rational function $S, \mathcal{F}(S)$, is the set of points $x \in \mathbb{R}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\mathbb{R}$ is the *Julia set*, $\mathcal{J}(S)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.