Applied Mathematics and Computation 248 (2014) 235–245

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Limit cycle bifurcations in a class of piecewise smooth systems with a double homoclinic loop ☆

Yuanyuan Liu [a,*], Valery G. Romanovski [a,b,c]

[a] Department of Mathematics, Shanghai Normal University, Shanghai, PR China
[b] Center for Applied Mathematics and Theoretical Physics, University of Maribor, Maribor, Slovenia
[c] Faculty of Natural Science and Mathematics, University of Maribor, Koroška 160, Maribor, Slovenia

ARTICLE INFO

Keywords:
Hamiltonian system
Double homoclinic loop
Melnikov function
Limit cycle

ABSTRACT

In this paper we consider a class of perturbed piecewise smooth systems. Applying the method of first order Melnikov function we give a lower bound for the maximal number of limit cycles bifurcated from a double homoclinic loop. As an application we construct a piecewise quadratic system with quartic perturbation, which has 11 limit cycles bifurcated from such loop.

1. Introduction and preliminaries

The bifurcation theory is an important part of the qualitative theory of differential systems. The study of bifurcations from singular points and periodic orbits is important for the analysis of many mathematical models, like, for instance, the predator–prey system, the Holling–Tanner model, the infection model (see [21,23,25] and the references therein), and many other models. The objects of the main interest in such models are usually isolated periodic orbits, since they describe auto-oscillating regimes of the system. It appears the main technique to study limit cycles is a perturbation of simple systems with annulus of periodic orbits, in particular, a perturbation of Hamiltonian systems. In the latter case an efficient tool used to estimate the number of limit cycles is the so-called Melnikov function, see e.g. [8,15] and references given there. The Melnikov function can be used to study the number of limit cycles bifurcated from a center, a homoclinic loop, a heteroclinic loop or an annulus consisting of a family of periodic orbit. For instance, the authors of [24] proved that generic planar quadratic Hamiltonian systems with the third degree polynomial perturbation can have eight small-amplitude limit cycles around a center. Roussarie [20] studied the following system

$$\begin{cases} \dot{x} = H_y + \varepsilon p(x,y,\varepsilon,\delta), \\ \dot{y} = -H_x + \varepsilon q(x,y,\varepsilon,\delta), \end{cases} \tag{1.1}$$

where $H(x,y), p(x,y,\varepsilon,\delta), q(x,y,\varepsilon,\delta)$ are analytic functions, $\varepsilon \geqslant 0$ is small and $\delta \in D \subset R^m$ is a vector parameter with D being a compact set. Under the assumption that the origin is a hyperbolic saddle, the author obtained the expansion of the Melnikov function $M(h,\delta)$ near the homoclinic loop L_0 as follows

$$M(h,\delta) = c_0(\delta) + c_1(\delta)h\ln|h| + c_2(\delta)h + c_3(\delta)h^2\ln|h| + O(h^2), \quad 0 < -h \ll 1.$$

☆ The project was supported by National Natural Science Foundation of China (11271261), the Slovenian Research Agency and by a Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme, FP7-PEOPLE-2012- IRSES-316338.

* Corresponding author.
E-mail addresses: liuyuanyuan1117@163.com (Y. Liu), valery.romanovsky@uni-mb.si (V.G. Romanovski).

http://dx.doi.org/10.1016/j.amc.2014.09.125
0096-3003/

Then the authors of [11] gave formulas for c_1 and c_2, and recently in [10] the formula for the coefficient c_3 has been obtained. The authors of [4] discussed bifurcations of periodic orbits of a class of planar systems with one switching line. They derived an expression of the first order Melnikov function and applied it to study the number of limit cycles bifurcated from the annulus.

To match the theoretic development and applications to models of real life phenomena, many researchers paid a lot of attention to the area of piecewise smooth system, see monographs [1,5,14], articles [3,6,7,19] and references therein. It is well-known that for smooth dynamical system (1.1) if Melnikov function $M(h,\delta)$ satisfies for some small $|h^*|>0$ the condition

$$M(h^*,\delta)=M'(h^*,\delta)=M''(h^*,\delta)=\cdots=M^{k-1}(h^*,\delta)=0,\quad M^k(h^*,\delta)\neq 0,$$

then the corresponding system has at most k limit cycles near L_{h^*} for $\varepsilon>0$ sufficient small, and has at least one limit cycle if k is odd, see Theorem 6.1 in [15]. A similar version for non-smooth systems was given in the paper [18] by Liu and Han. A number of new results on the problem are obtained also in [2,12,16,17]. The authors of [16,17] discussed the piecewise smooth systems with the origin as a center and a degenerated saddle (the definition can be found in [12]), respectively. Differently, we treat the origin as a hyperbolic saddle in the unperturbed system of (1.1).

In this paper, we consider a piecewise near-Hamiltonian system of the form (1.1), where $\varepsilon>0$ is small and $\delta\in D\subset R^m$ is a vector parameter, with D compact, and

$$H(x,y)=\begin{cases}H^+(x,y), & x\geqslant 0,\\ H^-(x,y), & x<0,\end{cases}$$

$$p(x,y,\varepsilon,\delta)=\begin{cases}p^+(x,y,\varepsilon,\delta), & x\geqslant 0,\\ p^-(x,y,\varepsilon,\delta), & x<0,\end{cases}$$

$$q(x,y,\varepsilon,\delta)=\begin{cases}q^+(x,y,\varepsilon,\delta), & x\geqslant 0,\\ q^-(x,y,\varepsilon,\delta), & x<0,\end{cases}$$

with $H^\pm,p^\pm,q^\pm$ being analytical functions defined on $\mathbb{R}^2$.

We call the analytic systems

$$\begin{cases}\dot{x}=H_y^+ +\varepsilon p^+(x,y,\varepsilon,\delta),\\ \dot{y}=-H_x^+ +\varepsilon q^+(x,y,\varepsilon,\delta),\end{cases}\tag{1.2}$$

$$\begin{cases}\dot{x}=H_y^- +\varepsilon p^-(x,y,\varepsilon,\delta),\\ \dot{y}=-H_x^- +\varepsilon q^-(x,y,\varepsilon,\delta),\end{cases}\tag{1.3}$$

the right subsystem and the left subsystem, respectively.

For $\varepsilon=0$, systems (1.1)–(1.3) become, respectively,

$$\begin{cases}\dot{x}=H_y,\\ \dot{y}=-H_x,\end{cases}\tag{1.4}$$

$$\begin{cases}\dot{x}=H_y^+,\\ \dot{y}=-H_x^+,\end{cases}\tag{1.5}$$

$$\begin{cases}\dot{x}=H_y^-,\\ \dot{y}=-H_x^-.\end{cases}\tag{1.6}$$

We will suppose that

H(I) The origin is a hyperbolic saddle for both systems (1.5) and (1.6), and the equations $H^\pm(x,y)=0$ for $\pm x\geqslant 0$ define two homoclinic loops $L_0^\pm$ with a critical point at the origin. See Fig. 1.1. Then L_0^+ and L_0^- form a double homoclinic loop $L_0=L_0^+\bigcup L_0^-$ in system (1.4).

Further, denoting by $L_h^\pm$ the orbits $\{H^\pm(x,y)=h,\pm x\geqslant 0, h\in(\alpha^\pm,\beta),\alpha^\pm<0<\beta\}$ we assume that for system (1.4).

H(II) There exist three families of periodic orbits $\{L_h^+|\alpha^+<h<0\},\{L_h^-|\alpha^-<h<0\}$ and $\{L_h|0<h<\beta\}$ where $L_h=\{L_h^+|0<h<\beta\}\bigcup\{L_{\widetilde{h}}^-|0<\widetilde{h}<\beta\}$. Note that the closed orbits $L_h^\pm|_{\alpha^\pm<h<0}$ are located inside the closed loop L_0 and $L_h^\pm|_{0<h<\beta}$ are two curves with the beginning and ending points being both at y-axis. Moreover, for $0<h<\beta$ we denote the orbit intersecting y-axis at $A(h)=(0,a(h))$ and $B(h)=(0,b(h))$ with $b(h)<0<a(h)$ and $a(0)=b(0)=0$ by L_h^+. Besides, we have

$$H^-(A(h))=H^-(B(h))=\widetilde{h}\tag{1.7}$$

for the definition of closed orbit L_h and denote $L_h|_{x\geqslant 0}$ and $L_h|_{x\leq 0}$ by $\widehat{AB}$ and $\widehat{BA}$, respectively. See Fig. 1.2.