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Two families of Bernstein–Durrmeyer type operators

Daniel Cárdenas-Morales [a,*], Vijay Gupta [b]

[a] Department of Mathematics, University of Jaén, Campus Las Lagunillas s/n., 23071 Jaén, Spain
[b] Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi 110078, India

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ABSTRACT

This paper deals with two families of Bernstein–Durrmeyer type operators that arise as integral modifications respectively from the classical Bernstein operators and from Bernstein-type operators based on the Polya distribution. An important role is played by the so-called genuine elements of each class, the only ones that reproduce linear functions. Asymptotic formulae and direct convergence results are stated. Finally some modifications of the sequences of the families are introduced, in such a way that the resulting operators reproduce linear functions and are comparable with the genuine sequences.

1. Introduction and preliminaries

Durrmeyer [7] and Lupaş [19] introduced independently the following modification of the classical Bernstein operators, which associates with each function f integrable on the interval $[0, 1]$ the polynomial

$$M_nf(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad x \in [0, 1], \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}.$$

Some further modifications that obey the following expression, for certain non negative integers $\alpha \geqslant \beta \geqslant 0$ and certain index set $I_n \subset \{0, 1, \ldots, n\}$, were studied later:

$$M_{n,\alpha,\beta}f(x) = \sum_{k\in I_n} p_{n,k}(x)f(k/n) + (n-\alpha+1)\sum_{k=\beta}^{n-\alpha+\beta} p_{n,k}(x)\int_0^1 p_{n-\alpha,k-\beta}(t)f(t)dt.$$

Thus, around 1987 Chen [4] and Goodman and Sharma [9] considered the special case $\alpha = 2, \beta = 1, I_n = \{0, n\}$, usually called the *genuine Bernstein–Durrmeyer operators*, which have been intensively studied in many articles (see for instance [22,23,26,8]; here $p_{0,0}(x) = 1$ and we use the general convention $p_{n,k}(x) = 0$ if $n, k \in \mathbb{N}$ do not satisfy the condition $0 \leqslant k \leqslant n$).

In 1997 Gupta [11] dealt with the case $\alpha = 1, \beta = 0, I_n = \{n\}$, and more recently Gupta and Maheshwari [16] and Abel et al. [1] introduced the cases $\alpha = 1, \beta = 1, I_n = \{0\}$ and $\alpha = 0, \beta = 1, I_n = \{0\}$.

* Corresponding author.
E-mail addresses: cardenas@ujaen.es (D. Cárdenas-Morales), vijaygupta2001@hotmail.com (V. Gupta).

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Obviously, the choice $\alpha = 0, \beta = 0, I_n = \emptyset$ corresponds to (1); they are the *classical Bernstein–Durrmeyer operators*, whose first study in depth was carried out by Derrienic [5].

One of the key elements of this paper is to notice that it also makes sense to consider the sequences of operators of the two-parameter family $M_{n,\alpha,\beta}$ when, beyond the aforesaid condition $\alpha \geqslant \beta \geqslant 0$, we let α and β be any elements from $\mathbb{Z}$ with $\alpha \leqslant 2$ and $\alpha - 1 \leqslant \beta \leqslant 1$, and $I_n = \{i : 0 \leqslant i < \beta\} \cup \{i : n - \alpha + \beta < i \leqslant n\}$. For all these values of α and β, by using the notation $a^+ = \max\{0, a\}, a \in \mathbb{R}$, the operators may be rewritten in the following manner:

$$M_{n,\alpha,\beta}f(x) = \beta^+ f(0)(1-x)^n + (n-\alpha+1)\sum_{k=\beta^+}^{n-(\alpha-\beta)^+} p_{n,k}(x)\int_0^1 p_{n-\alpha,k-\beta}(t)f(t)dt + (\alpha-\beta)^+ f(1)x^n. \tag{2}$$

Notice that $\beta^+ = \begin{cases} 1, & \text{if } \beta = 1; \\ 0, & \text{otherwise,} \end{cases}$ and $(\alpha-\beta)^+ = \begin{cases} 1, & \text{if } \beta = \alpha - 1; \\ 0, & \text{otherwise.} \end{cases}$

Moreover, if we denote by $\delta = \delta(t)$ the Dirac delta function, we can even rewrite the operators as

$$M_{n,\alpha,\beta}f(x) = \int_0^1 K_n^{\alpha,\beta}(x,t)f(t)dt,$$

where

$$K_n^{\alpha,\beta}(x,t) = \beta^+(1-x)^n\delta(t) + (n-\alpha+1)\sum_{k=\beta^+}^{n-(\alpha-\beta)^+} p_{n,k}(x)p_{n-\alpha,k-\beta}(t) + (\alpha-\beta)^+x^n\delta(t-1).$$

Our first aim with this paper is to study the sequences of operators of the family

$$\{M_{n,\alpha,\beta} : \ \alpha,\beta \in \mathbb{Z}, \quad \alpha \leqslant 2, \quad \alpha - 1 \leqslant \beta \leqslant 1\},$$

bringing at the same time a sort of unification to all those papers which have dealt with the aforementioned particular cases. Thus, in Section 2 we deal with the moments and derive an asymptotic formula, which extends known results for the aforementioned particular cases. As a byproduct, we see that all the operators of the family reproduce constants and only the genuine ones reproduce linear functions. In Section 3 we show that $M_{n,\alpha,\beta}f$ represents an approximation process towards f for $f \in C[0,1]$, and prove a direct estimate. Section 4 contains estimates of the rate of convergence for absolutely continuous functions whose first derivative coincides with a function of bounded variation on each subinterval of $(0,1)$.

As a second interesting point of this work, in Section 5 we introduce a modification of each sequence $M_{n,\alpha,\beta}$, $\alpha < 2$, in such a way that the resulting operators, say $\widetilde{M}_{n,\alpha,\beta}$, reproduce linear functions. We prove that the sequences $\widetilde{M}_{n,\alpha,\beta}$ present orders of approximation at least as good as the one of $M_{n,2,1}$ in the intervals where they make sense, namely $I_{\alpha,\beta} := \left[\frac{1-\beta}{3-\alpha}, \frac{2-\beta}{3-\alpha}\right]$. In connection with this, we point up that for $\alpha < 2$ and $\beta = 1, 0, -1, \dots, \alpha - 1$, the subintervals $I_{\alpha,\beta}$ cover the interval $[0,1]$, while for $\alpha = 2$ (and consequently $\beta = 1$) $I_{\alpha,\beta} = [0,1]$; this fact make us see the genuine Bernstein–Durrmeyer sequence as a limit element of certain one-parameter family of sequences of piecewise polynomial operators which reproduce linear functions.

Finally, in Section 6, we accomplish our last objective by proposing a parallel study for the integral modification of the following sequence of Bernstein-type operators based on the Polya distribution:

$$P_n^{(1/n)}(f,x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^n f\left(\frac{k}{n}\right)(nx)_k(n-nx)_{n-k},$$

where the rising factorials $(nx)_k = \prod_{i=0}^{k-1}(nx+i)$, $(n-nx)_{n-k} = \prod_{i=0}^{n-k-1}(n-nx+i)$ are used. This sequence was firstly considered by Lupaş and Lupaş [20] as a particular case of a general family introduced by Stancu in [25].

2. Moments and central moments

It is clear that each single operator $M_{n,\alpha,\beta}$ reproduces constants and produces algebraic polynomials of degree less than or equal to n. On the other hand, recall that the genuine Bernstein–Durrmeyer operators, $M_{n,2,1}$, reproduce linear functions as well, thus interpolating every function $f \in C[0,1]$ at the points 0 and 1.

The following results provide us with formulae for the moments and for the central moments. As usual, we use the notation $e_m(t) = t^m$, $e_m^x(t) = (t-x)^m$ and

$$\mu_{m,n}(x) = \mu_{m,n,\alpha,\beta}(x) = M_{n,\alpha,\beta}e_m^x(x).$$

Obviously,

$$M_{n,\alpha,\beta}e_0 = e_0, \quad \mu_{0,n}(x) = 1. \tag{3}$$

Proposition 1. *Let m, n be positive integers with $n > 1$ and $m \geqslant 1$. Then*

$$M_{n,\alpha,\beta}e_m(x) = \frac{(n-\alpha+1)!}{(n-\alpha+1+m)!}\sum_{k=\beta^+}^m \binom{m}{k}\frac{(m-\beta)!n!}{(k-\beta)!(n-k)!}x^k.$$