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On the Hilbert operator and the Hilbert formulas on the unit sphere for the time-harmonic Maxwell equations

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ABSTRACT

In this work we establish some analogues of the Hilbert formulas on the unit sphere for the theory of time-harmonic (monochromatic) electromagnetic fields. Our formulas relate one of the components of the limit value of a time-harmonic electromagnetic field in the unit ball to the rest of components. The obtained results are based on the close relation between time-harmonic solutions of the Maxwell equations and the three-dimensional α-hyperholomorphic function theory. Hilbert formulas for α-hyperholomorphic function theory for α being a complex number are also obtained, such formulas relate a pair of components of the boundary value of an α-hyperholomorphic function in the unit ball to the other pair of components, in an analogy with what happens in the case of the theory of functions of one complex variable.

1. Introduction

In the theory of functions of one complex variable, given a function holomorphic in the unit disk or in the upper half-plane what is called usually the Hilbert formulas is the relation between the real components of the boundary value of the holomorphic function; in other words, it is a relation between the boundary values of a pair of conjugate harmonic functions. An encyclopedic source of information about them is the book [2].

Given a limit function f on the unit circle $\mathbb{S}$, denote $g(\varphi) := f(e^{i\varphi}), 0 \leqslant \varphi \leq 2\pi$, and $g = g_1 + ig_2$. Then the real components g_1 and g_2 of g are related by the following formulas known as the Hilbert formulas for the unit disk, or for the unit circumference:

$$\begin{aligned} M[g_1] + H[g_2] &= g_1, \\ M[g_2] - H[g_1] &= g_2, \end{aligned} \tag{1}$$

where M and H are defined on the linear space $C^{0,\mu}(\mathbb{S}), 0 < \mu < 1$, by the following integrals:

$$H[g](\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\psi - \varphi}{2} g(\psi)d\psi, \quad \varphi \in [0, 2\pi], \tag{2}$$

$$M[g] := \frac{1}{2\pi}\int_0^{2\pi} g(\psi)d\psi. \tag{3}$$

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The integral $H[g]$ is understood in the sense of Cauchy's principal value, generating the so-called Hilbert operator with (real) kernel $\frac{1}{2\pi}\cot\frac{\psi-\varphi}{2}$ which is well-defined on $C^{0,\mu}(\mathbb{S}), 0<\mu<1$, and where M is a functional that can be seen as the average value of the function.

The Hilbert operator (2) is a well-known transformation in mathematics and in signal processing; for example, in geophysics and astrophysics it deals with input signals. Examples of this type of signals are seismic, satellite and gravitational data; and the Hilbert operator proves to be useful for a local analysis of them, providing a set of rotation-invariant local properties: the local amplitude, local orientation and local phase, see, e.g., [10] where the Hilbert formulas also have proven to be useful in techniques used to separate geophysical fields. Thus, the same kind of applications can be expected for our spatial analogue of the "plane" Hilbert operator.

In this work we establish some analogues of the Hilbert formulas (1) on the unit sphere for time-harmonic (monochromatic) electromagnetic fields in the unit ball. Our formulas relate one of the components of the limit value of a time-harmonic electromagnetic field in the unit ball to the rest of components; thus, their structure is deeply similar to that of their one-dimensional antecedents.

The obtained results are based on the close relation between time-harmonic solutions of the Maxwell equations and the three-dimensional α-hyperholomorphic function theory following the approach presented in [3], see also [8,9]. Hilbert formulas for α-hyperholomorphic function theory for α being a complex number are also obtained, such formulas relate a pair of components of the boundary value of an α-hyperholomorphic function in the unit ball to the other pair of components.

In [5] the authors established some analogues of the Hilbert formulas on the unit sphere for the case of solenoidal and irrotational vector fields in the unit ball meanwhile in [6] the analogues are presented for the time-harmonic relativistic Dirac bispinors theory.

Works in this line but on the case of a half space have been done in [7,3, p. 142–144] and [10, p. 253–2-55] where the obtained integral transforms are referred to as Stratton–Chu integral transforms.

The paper is organized as follows. In Section 2 the analogues of the Hilbert formulas on the unit sphere for time-harmonic electromagnetic fields in the unit ball are announced and their corollaries are formulated; the proof of the Hilbert formulas is given in Section 6 and is based in the contents of Sections 4 and 5. Then, in Section 3 we present a brief review of the α-hyperholomorphic function theory. Section 4 presents the Hilbert formulas and their proof in the context of quaternionic analysis for the theory of α-hyperholomorphic functions for a complex number α which relate a pair of components of the boundary value of a quaternionic hyperholomorphic function in the unit ball to the other pair of components. Section 5 contains a description of the relations between the time-harmonic electromagnetic fields theory and the theory of α-hyperholomorphic functions. Finally, in Section 6 we obtain the results announced in Section 2 using what is proved in Section 5.

2. The Hilbert formulas on the unit sphere for the time-harmonic electromagnetic fields theory

In the first part of this section we will describe the approach presented in [3], see also [8,9]. Let Ω be a domain in $\mathbb{R}^3$, let Γ be its boundary, and let $\vec{E},\ \vec{H}:\Omega\subset\mathbb{R}^3\to\mathbb{C}^3$ be a pair of complex-valued vector fields. The time-harmonic (monochromatic) Maxwell equations are defined by the following system

$$\mathrm{rot}\vec{H}=\sigma\vec{E},\quad \mathrm{rot}\vec{E}=i\omega\mu\vec{H}, \tag{4}$$

$$\mathrm{div}\vec{H}=0,\quad \mathrm{div}\vec{E}=0, \tag{5}$$

where $\sigma:=\sigma^*-i\omega\varepsilon$ is the complex electrical conductivity; ε is the dielectric constant; μ is the magnetic permeability; σ^* is the medium electrical conductivity being inverse to its electrical resistivity: $\sigma^*=\frac{1}{\rho}$. It is assumed that Ω is filled up with a homogeneous medium and there are no currents and charges in Ω.

If $\vec{E}$ and $\vec{H}$ form a solution to the system (4) and (5) in Ω, then $(\vec{E},\vec{H})$ is called a time-harmonic (monochromatic) electromagnetic field. Moreover, $\vec{E}$ and $\vec{H}$ are called, respectively, the electrical and magnetic components of the electromagnetic field in Ω and they satisfy the homogeneous Helmholtz equation:

$$\Delta\vec{E}+\lambda\vec{E}=0, \tag{6}$$

$$\Delta\vec{H}+\lambda\vec{H}=0, \tag{7}$$

where $\lambda:=i\omega\mu\sigma^*+\omega^2\mu\varepsilon=i\omega\mu\sigma\in\mathbb{C}$ and its square root $\alpha:=\sqrt{\lambda}$ is called a medium wave number.

Notice that the equalities (5) can be deduced directly form (4) but there are some substantial mathematical reasons (see [8,9]) to write them down explicitly. Also, it is worth mentioning that it is known (see [1], e.g.) that for any solution $(\vec{E},\vec{H})$ of the system (4) and (5) its components satisfy:

$$\langle\vec{E},\vec{H}\rangle=0, \tag{8}$$

where $\langle\vec{E},\vec{H}\rangle:=\sum_{k=1}^{3}E_kH_k$ is the inner product that determines a $\mathbb{C}$-valued bilinear form over $\mathbb{C}$.