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Fault-tolerance of (n,k)-star networks ☆

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ABSTRACT

This paper considers a refined measure $\kappa_s^{(h)}$ for the fault-tolerance of a network and, for the generalized star network $S_{n,k}$, determines $\kappa_s^{(h)}(S_{n,k}) = n + h(k-2) - 1$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$, which implies that at least $n + h(k-2) - 1$ vertices of $S_{n,k}$ have to be removed to get a disconnected graph without vertices of degree less than h. This work generalizes some known results. When the (n,k)-star graph is used to model the topological structure of a large-scale parallel processing system, this result can provide a more accurate measure for the fault tolerance of the system.

1. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph in which vertices correspond to processors and edges correspond to communication links.

The *connectivity* $\kappa(G)$ of a graph G is defined as the minimum number of vertices whose deletion disconnects G. As an important measure for the fault-tolerance of a network, the larger connectivity κ is, the more reliable the network is. However, the definition of κ is implicitly assumed that any subset of system components is equally likely to be faulty simultaneously, which may not be true in real applications, thus connectivity κ underestimate the reliability of a network. To compensate such shortcoming, Harary [12] introduced the concept of the conditional connectivity by appending some requirements on the resulting graph. In this trend, Esfahanian [11] proposed the concept of the restricted connectivity, Latifi et al. [16] generalized it to the restricted h-connectivity which can measure fault tolerance of an interconnection network more accurately than the classical connectivity κ. The concepts stated here are slightly different from theirs.

For a given nonnegative integer h, a subset S of vertices of a connected graph G is called an *h-super vertex-cut*, or *h-cut* for short, if $G - S$ is disconnected and has the minimum degree at least h. The *h-super connectivity* of G, denoted by $\kappa_s^{(h)}(G)$, is defined as the minimum cardinality over all h-cuts of G. Since a complete graph K_n is nonseparable, $\kappa_s^{(h)}(K_n)$ does not exist for any h with $0 \leqslant h \leqslant n-1$. Furthermore, if G is not a complete graph then $\kappa_s^{(0)}(G) = \kappa(G)$; for $h \geqslant 1$, if $\kappa_s^{(h)}(G)$ exists, then $\kappa_s^{(h-1)}(G) \leqslant \kappa_s^{(h)}(G)$. For any graph G and integer h, determining $\kappa_s^{(h)}(G)$ is quite difficult. In fact, the existence of $\kappa_s^{(h)}(G)$ is an open problem so far when $h \geqslant 1$. Only a little knowledge of results have been known on $\kappa_s^{(h)}$ for particular classes of graphs and small h's.

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As a topological structure of interconnection networks, the star graph S_n, proposed by Akers and Krishnamurthy [1], is an attractive alternative to the hypercube as an interconnection network, and has superior degree and diameter compared to the comparable hypercube as well as it is highly hierarchical and symmetrical [9]. However, the number of vertices of an n-dimensional star is $n!$, there is a large gap between $n!$ and $(n+1)!$ if S_n is extended to S_{n+1}. To achieve scalability, Chiang and Chen [7] generalized the star graph S_n to the (n,k)-star graph $S_{n,k}$, which preserves many ideal properties of the star graph [8]. Since then the (n,k)-star graph has received considerable attention in the literature [2,3,5,6,4,10,14,15,19,18,22,24–27].

This paper is concerned about $\kappa_s^{(h)}$ for the (n,k)-star graph $S_{n,k}$. For $k = n-1$, $S_{n,n-1}$ is isomorphic to a star graph S_n, Hu and Yang [13], Nie et al. [20] and Rouskovet al. [21], independently, determined $\kappa_s^{(1)}(S_n) = 2n-4$ for $n \geqslant 3$. Wan and Zhang [23] showed $\kappa_s^{(2)}(S_n) = 6n-18$ for $n \geqslant 4$. Yang et al. [26] proved that if $2 \leqslant k \leqslant n-2$ then $\kappa_s^{(1)}(S_{n,k}) = n+k-3$ for $n \geqslant 3$ and $\kappa_s^{(2)}(S_{n,k}) = n+2k-5$ for $n \geqslant 4$.

We, in this paper, will generalize these results by proving that $\kappa_s^{(h)}(S_{n,k}) = n+h(k-2)-1$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$.

The main proof of this result is in Section 3. In Section 2, we recall the structure of $S_{n,k}$ and some lemmas used in our proofs. Conclusions and some remarks are in Section 4.

2. Definitions and lemmas

For a given integer n with $n \geqslant 2$, let $I_n = \{1, 2, \ldots, n\}$, $I'_n = \{2, \ldots, n\}$. For an integer k with $1 \leqslant k \leqslant n-1$, let $P(n,k) = \{p_1p_2\ldots p_k :\ p_i \in I_n,\ p_i \neq p_j,\ 1 \leqslant i \neq j \leqslant k\}$, the set of k-permutations on I_n. Clearly, $|P(n,k)| = \frac{n!}{(n-k)!}$.

Definition 2.1. (Chiang et al. [7]) The (n,k)-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$. The adjacency is defined as follows: a vertex $p = p_1p_2\ldots p_i\ldots p_k$ is adjacent to a vertex

(a) $p_ip_2\ldots p_{i-1}p_1p_{i+1}\ldots p_k$, where $i \in I'_k$ (swap p_1 with p_i).
(b) $p'_1p_2p_3\ldots p_k$, where $p'_1 \in I_n \setminus \{p_i :\ i \in I_k\}$ (replace p_1 by p'_1).

The vertices of type (a) are referred to as *swap-neighbors* of the vertex p and the edges between them are referred to as *swap-edges* or *i-edges*. The vertices of type (b) are referred to as *unswap-neighbors* of the vertex p and the edges between them are referred to as *unswap-edges*. Clearly, every vertex in $S_{n,k}$ has $k-1$ swap-neighbors and $n-k$ unswap-neighbors. Usually, if $p = p_1p_2\ldots p_k$ is a vertex in $S_{n,k}$, we call p_i the ith *bit* of p for each $i \in I_k$.

It has been known that the (n,k)-star graph $S_{n,k}$ is a vertex transitive graph with order $\frac{n!}{(n-k)!}$ and regular degree $n-1$ (see Chiang et al. [7]). In addition, $S_{n,n-1}$ is isomorphic to the star graph S_n, and $S_{n,1}$ is isomorphic to the complete graph K_n. Fig. 1 shows the $(4,2)$-star $S_{4,2}$ and the $(4,3)$-star $S_{4,3}$.

Lemma 2.2. *For any $\alpha = p_2p_3\ldots p_k \in P(n,k-1)$ $(k \geqslant 2)$, let $V_\alpha = \{p_1\alpha :\ p_1 \in I_n \setminus \{p_i :\ i \in I'_k\}\}$. Then the subgraph of $S_{n,k}$ induced by V_α is a complete graph of order $n-k+1$, denoted by K^α_{n-k+1}.*

Proof. For any two vertices $p_1\alpha$ and $p'_1\alpha$ in V_α with $p_1 \neq p'_1$, by the condition (b) of Definition 2.1, $p_1\alpha$ and $p'_1\alpha$ are linked in $S_{n,k}$ by an unswap-edge. Thus, the subgraph of $S_{n,k}$ induced by V_α is a complete graph K_{n-k+1}. $\square$

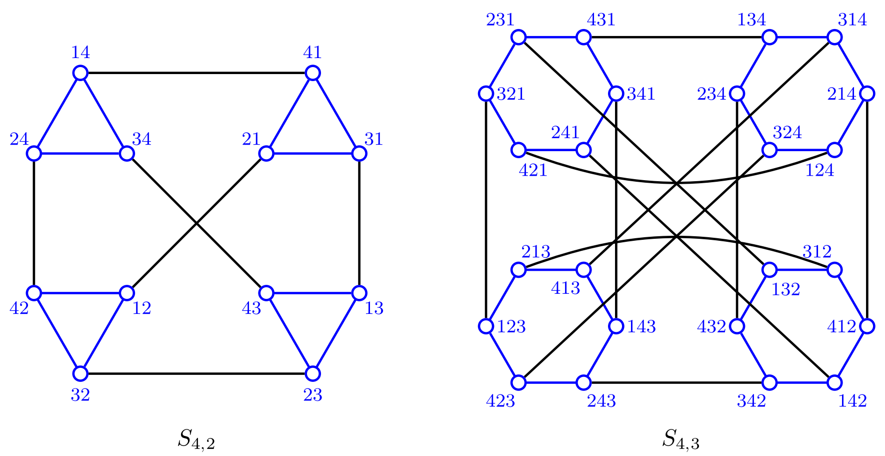

Fig. 1. The (4,2)-star $S_{4,2}$ and the (4,3)-star $S_{4,3}$