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Application of fractional order theory of thermoelasticity to a 2D problem for a half-space

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ABSTRACT

In this work, we apply the fractional order theory of thermoelasticity to a 2D problem for a half-space. The surface of the half-space is taken to be traction free and is subject to heating. There are no body forces or heat sources affecting the medium. Laplace and exponential Fourier transform techniques are used to solve the problem. The inverse Laplace transforms are obtained using a numerical technique.

The predictions of the theory are discussed and compared with those for the generalized theory of thermoelasticity. We also study the effect of the fractional derivative parameter on the behavior of the solution. Numerical results are computed and represented graphically for the temperature, displacement and stress distributions.

1. Introduction

Lord and Shulman [1] introduced the theory of generalized thermoelasticity with one relaxation time by using the Maxwell–Cattaneo law of heat conduction instead of the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. This theory was extended to thermoelastic diffusion in [2], to thermoviscoelasticity in [3] and to micropolar media in [4]. Exact solution for a problem of a spherical cavity was obtained by Sherief and Saleh [5]. Some problems for a penny shaped crack and a mode I crack were solved by Sherief and El-Maghraby [6,7].

Fractional calculus has been used successfully to modify many existing models of physical processes. Caputo and Mainardi [8,9] and Caputo [10] found good agreement with experimental results when using fractional derivatives for description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity.

The solution obtained by using ordinary derivatives predicts an instantaneous response while that obtained by using fractional derivatives predicts a retarded response that depends on the history of the applied causes. This is more in accord with physical observations [11].

The general space–time-fractional heat conduction equation in the one-dimensional case has been formulated by Gorenflo et al. [12]. Povstenko [13] made a review of thermoelasticity that uses fractional heat conduction equation. The theory of thermal stresses based on the heat conduction equation with the Caputo time-fractional derivative is used by Povstenko [14] to investigate thermal stresses in an infinite body with a circular cylindrical hole. Povstenko proposed and investigated new models that use fractional derivative in [15,16].

The fractional order theory of thermoelasticity was derived by Sherief et al. [17]. It is a generalization of both the coupled and the generalized theories of thermoelasticity. Sherief and Abd El Latief [18,19] have solved a 1D problems for a half space

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and for spherical cavity in this theory, studied the effect of the fractional derivative parameter on fractional thermoelastic material with variable thermal conductivity [20] and applied this theory to a 1D problem for a half-space overlaid by a thick layer of a different materials [21].

Ezzat [22] introduced a heat conduction law with time-fractional derivative to formulate a mathematical model of magneto-thermoelasticity theory. Ezzat and El-Karamany [23] solved a one-dimensional application for a conducting half-space by using a fractional mathematical model of magneto-thermoelasticity.

2. Formulation of the Problem

We consider a homogeneous isotropic thermoelastic solid occupying the half-space $y \geqslant 0$. The y-axis is taken perpendicular to the bounding plane pointing inward. We also assume that the initial state of the medium is quiescent. The surface of this medium is traction free and subject to heating on the surface of intensity $r(x,t)$.

Due to the physics of the problem, all relevant quantities will be functions of x, y and t only. The displacement vector $\mathbf{u}$ will have two components only u, v in the x and y directions, respectively.

The equation of motion, in the absence of body forces, has the form [17]

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\,\mathrm{grad}\, e + \mu \nabla^2 \mathbf{u} - \gamma\, \mathrm{grad}\, T, \tag{1}$$

where ρ is the density, λ, μ are Lamé's constants, T is the absolute temperature and γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$ where α_t is the coefficient of linear thermal expansion. ∇^2 is the two-dimensional Laplace's operator in Cartesian coordinates. e is the cubical dilatation, given by

$$e = \mathrm{div}\,\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{2}$$

The equation of energy, in the absence of heat sources, can be written as [17]

$$k \nabla^2 T = \frac{\partial}{\partial t}\left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right)(\rho C_E T + \gamma T_0 e) \tag{3}$$

C_E is the specific heat at constant strain, τ_0 is a constant, k is the thermal conductivity. The time fractional derivative of order α used is taken to be in the sense of Caputo fractional derivative, $0 \leqslant \alpha \leqslant 1$. T_0 is a reference temperature assumed to be such that $|(T - T_0)/T_0| \ll 1$.

These equations are supplemented by the constitutive equations [17]

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + (\lambda e - \gamma(T - T_0))\delta_{ij} \tag{4}$$

where σ_{ij} are the components of the stress tensor and δ_{ij} is the Kronecker delta.

We shall use the following non-dimensional variables:

$$X' = c\eta x \quad y' = c\eta y \quad t' = c^2 \eta t \quad u' = c\eta u,$$

$$v' = c\eta v \quad \tau_0' = c^{2\alpha}\eta^\alpha \tau_0 \quad \theta = \frac{\gamma(T - T_0)}{\lambda + 2\mu} \quad \sigma_{ij}' = \frac{\sigma_{ij}}{\mu},$$

where $\eta = \rho C_E / k$, $c = \sqrt{(\lambda + 2\mu)/\rho}$.

In terms of the above non-dimensional variables, Eq. (2) retains its form while Eqs. (1), (3) and (4) take the forms (dropping the primes for convenience)

$$\beta^2 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\beta^2 - 1)\,\mathrm{grad}\, e + \nabla^2 \mathbf{u} - \beta^2\, \mathrm{grad}\, \theta \tag{5}$$

$$\nabla^2 \theta = \frac{\partial}{\partial t}\left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right)(\theta + \varepsilon e), \tag{6}$$

$$\sigma_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + ((\beta^2 - 2)e - \beta^2 \theta)\delta_{ij}, \tag{7}$$

where $\beta^2 = (\lambda + 2\mu)/\mu$ *and* $\varepsilon = T_0 \gamma^2 / [\rho C_E (\lambda + 2\mu)]$.

Eq. (5) has two components:

$$\beta^2 \frac{\partial^2 u}{\partial t^2} = (\beta^2 - 1)\frac{\partial e}{\partial x} + \nabla^2 u - \beta^2 \frac{\partial \theta}{\partial x}, \tag{8}$$