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A new method based on generalized Taylor expansion for computing a series solution of the linear systems

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ABSTRACT

In this paper, based on the generalized Taylor expansion and using the iteration matrix G of the iterative methods, we introduce a new method for computing a series solution of the linear systems. This method can be used to accelerate the convergence of the basic iterative methods. In addition, we show that, by applying the new method to a divergent iterative scheme, it is possible to construct a convergent series solution and to find the convergence intervals of control parameter for special cases. Numerical experiments are given to show the efficiency of the new method.

1. Introduction

Consider the linear system of equations,

$$Au = d, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is given, $d \in \mathbb{R}^n$ is known, and $u \in \mathbb{R}^n$ is unknown. One class of iterative methods is based on a splitting (M, N) of the matrix A, i.e.,

$$A = M - N, \tag{1.2}$$

where M is taken to be invertible and cheap to invert, meaning that a linear system with matrix coefficient M is much more economical to solve than (1.1). Based on (1.2), (1.1) can be written in the fixed-point form

$$u = Gu + c, \quad G = M^{-1}N, \quad c = M^{-1}d; \tag{1.3}$$

which yields the following iterative scheme for the solution of (1.1):

$$u^{(k+1)} = Gu^{(k)} + c, k = 0, 1, 2, \ldots, \quad u^{(0)} \in \mathbb{R}^n \text{ is arbitrary}. \tag{1.4}$$

There have been many studies about the convergence of the splitting iteration method (1.4), or in other words, the matrix splitting (1.2), when the coefficient matrix A and the iteration matrix G have particular properties (see, e.g. [1,3,4,13–15]). Also, for improving the rate of the convergence, many authors introduced the pereconditioned methods, see [5,7,11,12,16,17]. A sufficient and necessary condition for (1.4) to converge to the solution of (1.1), is that $\rho(G) < 1$, where $\rho(G)$ denotes the spectral radius of the iteration matrix G. In this paper based on the generalized Taylor expansion and using

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the iteration matrix G of the iterative method, we introduce a new method for computing a series solution of the linear system (1.1). We show that, this method can be used to accelerate the convergence of the convergent basic iterative methods. In addition, we prove that, under certain assumptions, this method can be applied to a divergent iterative scheme and to construct a convergent series solution.

This paper is organized as follows. In Section 2, by using generalized Taylor expansion we introduce the new method. In Section 3, we derive the conditions for improving the rate of convergence of the basic iterative methods. In Section 4, we apply the new method to the divergent iterative methods to construct a convergent series solution. We derive the convergence intervals and obtain the optimal value for the control parameter. In Section 5, some numerical examples are presented to show the efficiency of the method. Finally, we make some concluding remarks in Section 6.

2. New method based on generalized Taylor expansion

In the book [8], Liao introduces the generalized Taylor expansion to the nonlinear equation as follow,

$$f(t)=\lim_{m\to\infty}\sum_{l=0}^{m}\mu_0^{m,l}(\hbar)\frac{f^{(l)}(t_0)}{l!}(t-t_0)^l, \tag{2.1}$$

where $\lim_{m\to\infty}\mu_0^{m,l}(\hbar)=1$, for $l>1$. He controlled the convergence region of the generalized Taylor expansion (2.1) by the auxiliary parameter $\hbar\neq 0$. As [9], we call it the convergence control parameter $\hbar$.

In [9], through detailed analysis of some examples, Liu showed that the generalized Taylor series is only the usual Taylor expansion at point t_0. Here, by using this idea, we consider the Taylor expansion of $f(t)=\frac{1}{1-t}$ at point t_0

$$f(t)=\frac{1}{1-t_0}\left[1+\frac{t-t_0}{1-t_0}+\left(\frac{t-t_0}{1-t_0}\right)^2+\cdots+\left(\frac{t-t_0}{1-t_0}\right)^l+\cdots\right],$$

with the convergence region $|t-t_0|<|1-t_0|$. If we take $t_0=\alpha(1+\frac{1}{\hbar})$ where $\alpha=a+ib$, so the above expression becomes

$$f(t)=\frac{\hbar}{\hbar-\alpha(\hbar+1)}\left[1+\left(\frac{\hbar t-\alpha(\hbar+1)}{\hbar-\alpha(\hbar+1)}\right)+\left(\frac{\hbar t-\alpha(\hbar+1)}{\hbar-\alpha(\hbar+1)}\right)^2+\cdots+\left(\frac{\hbar t-\alpha(\hbar+1)}{\hbar-\alpha(\hbar+1)}\right)^l+\cdots\right]. \tag{2.2}$$

By assuming $\lambda_i\neq 1$, for $i=1,2,\ldots,n$, where λ_i is the eigenvalue of matrix G, and $\rho\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)<1$, if we apply (2.2) to the equation $f(G)=(I-G)^{-1}$, then one may obtain the following equation

$$f(G)=\frac{\hbar}{\hbar-\alpha(\hbar+1)}\left[I+\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)+\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)^2+\cdots+\left(\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}\right)^l+\cdots\right], \tag{2.3}$$

Taking

$$G_{\alpha,\hbar}=\frac{\hbar G-\alpha(\hbar+1)I}{\hbar-\alpha(\hbar+1)}, \tag{2.4}$$

yields

$$f(G)=\frac{\hbar}{\hbar-\alpha(\hbar+1)}[I+G_{\alpha,\hbar}+G_{\alpha,\hbar}^2+\cdots+G_{\alpha,\hbar}^l+\cdots]. \tag{2.5}$$

Let u_0 be an initial approximation to the exact solution u of the original system (1.1) and define the vectors

$$\begin{aligned} u_1 &= -\tfrac{\hbar}{\hbar-\alpha(\hbar+1)}[(I-G)u_0-c],\\ u_i &= G_{\alpha,\hbar}u_{i-1},\quad i=2,3,\ldots, \end{aligned} \tag{2.6}$$

and

$$v=\sum_{i=0}^{\infty}u_i=u_0+\sum_{i=1}^{\infty}G_{\alpha,\hbar}^{i-1}u_1 \tag{2.7}$$

It is obvious that if $\rho(G_{\alpha,\hbar})<1$, then the series $\sum_{i=1}^{\infty}G_{\alpha,\hbar}^{i-1}u_1$ converges and we have

$$v=u_0+(I-G_{\alpha,\hbar})^{-1}u_1=u_0+\left(\frac{\hbar}{\hbar-\alpha(\hbar+1)}\right)^{-1}(I-G)^{-1}u_1=(I-G)^{-1}c,$$

which is the exact solution of (1.3). So, when $\rho(G_{\alpha,\hbar})<1$ and $\lambda_i\neq 1$, for $i=1,2,\ldots,n$, a series of vectors can be computed by (2.6) and the convergence series (2.7) provides the following approximations to the exact solution of (1.1):

$$v_l=\sum_{i=0}^{l}u_i,\quad l=1,2,\ldots. \tag{2.8}$$

In this paper, our aim is to choose the convergence control parameter $\hbar\neq 0$ and appropriate $a+ib=\alpha\in\mathbb{C}$, so that $\rho(G_{\alpha,\hbar})<1$.