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Inequalities and asymptotic expansions for the constants of Landau and Lebesgue

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ABSTRACT

The constants of Landau and Lebesgue are defined, for all integers $n \geqslant 0$, in order, by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)} \right| \mathrm{d}t,$$

which play important roles in the theories of complex analysis and Fourier series, respectively. Diverse inequalities and approximations for these constants have been investigated and developed by many authors. Here, in this paper, we establish new asymptotic expansions for the constants G_n and $L_{n/2}$ of Landau and Lebesgue, respectively, in terms of the digamma and polygamma functions. Based on our expansion for the Landau constants G_n, we present new bounds for the Landau constants G_n in terms of the digamma and polygamma functions. We also establish inequalities for the Lebesgue constants $L_{n/2}$, which are applied to derive an asymptotic expansion for $L_{n/2}$ in terms of $1/(n+1)$. Furthermore, by giving numerical calculations to be compared, among several developed asymptotic expansions for the constants G_n and $L_{n/2}$, it is shown that our expansions presented here would be best ones.

1. Introduction

The Landau constants are defined by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which play an important role in the theory of complex analysis. More precisely, in 1913, Landau [21] proved that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function in the unit disc $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{C}$ being the set of complex numbers, which satisfies $|f(z)| < 1$ for all $z \in \mathcal{D}$, then the following optimal bounds hold true:

$$\left| \sum_{k=0}^{n} a_k \right| \leqslant G_n \quad (n \in \mathbb{N}_0). \tag{1.2}$$

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The Lebesgue constants are defined by

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}\right|\mathrm{d}t \quad (n \in \mathbb{N}_0), \tag{1.3}$$

which play an important role in the theory of Fourier series. More precisely, in 1906, Lebesgue [22] proved the following result: Assume a function f is integrable on the interval $[-\pi, \pi]$ and $S_n(f, x)$ is the nth partial sum of the Fourier series of f. That is,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\mathrm{d}t (k \in \mathbb{N}_0) \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\mathrm{d}t \quad (k \in \mathbb{N})$$

and

$$S_n(f,x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx)) \quad (n \in \mathbb{N}_0),$$

where the empty sum is (as usual, throughout this paper) understood to be nil. If $|f(x)| \leqslant 1$ for all $x \in [-\pi, \pi]$, then

$$S_n(f,x) \leqslant L_n \quad (n \in \mathbb{N}_0). \tag{1.4}$$

It is noted that L_n is the smallest possible constant for which the inequality (1.4) holds true for all continuous functions f on $[-\pi, \pi]$.

Diverse inequalities and approximations for the constants of Landau and Lebesgue have been investigated and developed by many authors. In this paper, we establish new asymptotic expansions for the constants G_n and $L_{n/2}$ of Landau and Lebesgue, respectively, in terms of the digamma and polygamma functions. Based on our expansion for the Landau constants G_n here, we present new bounds for the Landau constants G_n in terms of the digamma and polygamma functions. We also establish inequalities for the Lebesgue constants $L_{n/2}$, which are shown to be applied to derive an asymptotic expansion for $L_{n/2}$ in terms of $1/(n+1)$. Furthermore, by giving numerical calculations to be compared, among several developed asymptotic expansions for the constants G_n and $L_{n/2}$, it is shown that our expansions presented here would be best ones.

For this purpose, we begin by recalling the familiar (Euler's) gamma function $\Gamma(z)$ defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t \quad (\Re(z) > 0), \tag{1.5}$$

which is one of the simplest and most important special functions and has several other important equivalent forms (see, e.g., [32, Section 1.1]), knowledge of whose properties is a prerequisite for the study of many other special functions. The gamma function $\Gamma(z)$ arises in many areas of mathematics such as applied mathematics as well as mathematical analysis. The origin, history, and development of the gamma function $\Gamma(z)$ are described very nicely by Davis [14].

The logarithmic derivative of the gamma function $\Gamma(z)$:

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\{\ln\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln\Gamma(z) = \int_1^z \psi(t)\mathrm{d}t \tag{1.6}$$

is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(z)$:

$$\psi^{(n)}(z) := \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\psi(z)\} \quad (n \in \mathbb{N}) \tag{1.7}$$

are called the polygamma functions. In particular, the functions $\psi'(z)$ and $\psi^{(2)}(z)$ are called the trigamma and tetragamma functions (see, e.g., [1, p. 260]).

The following lemma is required in the sequel.

Lemma 1.1. ([18,34]). *Brouncker found the following remarkable continued fraction formula:*

$$\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 = \cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n+\ddots}}}} \quad (n \in \mathbb{N}_0). \tag{1.8}$$

Very recently, Granath [18] derived the asymptotic expansions for the Landau constants (1.1) and related inequalities by using Brouncker's continued fraction formula (1.8).

From (1.8), it is easy to find the following inequality (see, cf., [18, pp. 741–742]):

$$\cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n}}}} < \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 < \cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n}}} \quad (n \in \mathbb{N}),$$