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# Cell-average multiresolution based on local polynomial regression. Application to image processing ☆

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A B S T R A C T

In Harten (1996) [32] presented a general framework about multiresolution representation based on four principal operators: decimation and prediction, discretization and reconstruction. The discretization operator indicates the nature of the data. In this work the pixels of a digital image are obtained as the average of a function in some defined cells. A family of Harten cell-average multiresolution schemes based on local polynomial regression is presented. The stability is ensured by the linearity of the operators obtained and the order is calculated. Some numerical experiments are performed testing the accuracy of the prediction operators in comparison with the classical linear and nonlinear methods.


## 1. Introduction

In the last years, various techniques based on multiscale decompositions have been used in signal and image processing [42,19]. A multiscale transformation allows for an efficient representation of the image data since a significant number of the transform coefficients (details) are small and can be quantized or discarded with little loss of real information [25,17]. Harten's multiresolution framework (MR) [31,32] generalizes wavelet transforms. It consists on two operators, *decimation* and *prediction*, which connect two levels of resolution. Many have worked to design Harten's MR decimation operators:

- Aràndiga et al. [14] presented some MR schemes based on hat-average discretization operator;
- Greteuer and Meyer [29] showed how to construct MR schemes for any finite impulse response decimation filter (included point-value, cell-average, hat-average);

and to construct Harten's MR prediction operators:

- Harten [31,32] developed the essentially nonoscillatory, ENO, (see e.g. [33]) MR schemes.
- Cohen et al. [21] studied the properties of ENO prediction as the convergence;
- Aràndiga et al. [9] presented the weighted ENO (see e.g. [37]) MR schemes;
- Amat et al. [3,2] introduced the piecewise polyharmonic (PPH) interpolation as prediction operator;
- Aràndiga et al. [11] used Learning statistical techniques to construct a prediction operator.

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Also, certain properties as the stability of these families of schemes have been studied (see, e.g., [30,22,6,13,4]). Moreover, Harten's MR has been used in signal and image processing (see, e.g., [8,15,10]).

Different setting can be considered depending on the nature of the data as point-value. For the treatment of real images, typically, it is used cell-average discretization (CA). This approach is developed to be better adapted than point-value, due to the fact that averaging is more robust to noise than point evaluation.

In [16] a family of prediction operators based on Local polynomial regression (LPR) [24,44,41,7,39,43,45,46,38,18,47,28,35,40] in point-value context is presented. This article is devoted to design a MR scheme for cell-average using LPR method.

The stability of the method is proved and stability explicit bounds are presented for $\ell^p$-norm with $p = 1, 2, \infty$. The order of the schemes is calculated. This method generalizes the approach based on interpolation (see, e.g., [12]).

This paper is organized as follows. We review the MR schemes in cell-average context. Afterwards, we design a family of prediction operators using LPR. In Section 4 some properties of these schemes are proved. Finally, we perform some numerical examples in compression image context.

## 2. Harten's MR transforms in the cell-average setting

There exist many different approaches to MR decompositions which are closely connected: wavelet bases, subband filtering, hierarchical splitting of finite element spaces. Here, it will be convenient to use the discrete framework of Harten, based on *decimation and prediction*, which we briefly recall below (more details can be found in [12] or [32]).

From a set of discrete data $f^k = (f_j^k)_{j=1}^{J_k}$, where $k$ represents the level of discretization, the *decimation operator* $\mathcal{D}_k^{k-1}$ computes $f^{k-1} = (f_j^{k-1})_{j=1}^{J_{k-1}}$, at the next coarser level of discretization ($2J_{k-1} = J_k$). The *prediction operator* $\mathcal{P}_{k-1}^k$ maps a coarse vector $f^{k-1}$ onto a finer one $\tilde{f}^k = (\tilde{f}_j^k)_{j=1}^{J_k}$, which should be thought as an approximation of $f^k$. In contrast to decimation, the prediction operator need not be linear, but should at least satisfy the consistency requirement

$$\mathcal{D}_k^{k-1}\mathcal{P}_{k-1}^k = \mathcal{I}. \tag{1}$$

Assuming that $f^k$ is viewed as cell averages, i.e. if $c_j^k = [x_{j-1}^k, x_j^k]$ then

$$f_j^k = \frac{1}{h_k}\int_{c_j^k} f(x)dx,$$

$x_j^k = jh_k$ and $h_k = 1/J_k$, of a certain function the decimation operator is defined by the half sum

$$(\mathcal{D}_k^{k-1}f^k)_j = \frac{1}{2}(f_{2j}^k + f_{2j-1}^k) = f_j^{k-1}.$$

The details can be simply defined by the prediction error at the odd samples i.e. $d_j^{k-1} = f_{2j-1}^k - \tilde{f}_{2j-1}^k$. Therefore, we can represent $f^k$ by $(f^{k-1}, d^{k-1})$.

By iteration of this process from $k = L$ to $k = 1$, we obtain a *multiscale decomposition* of $f^L$ into $(f^0, d^1, \dots, d^L)$.

In this context, a classical prediction technique is to construct on each interval a polynomial

$$z(x) = \sum_{m=0}^{s} a_m x^m,$$

of degree $r = s$ (even), which interpolates the cell-averages on some stencil $\mathcal{S} = \left\{x_{j-\frac{s}{2}-1}^{k-1}, \dots, x_{j+\frac{s}{2}}^{k-1}\right\}$ containing this interval, i.e. such that

$$\frac{1}{h_{k-1}}\int_{c_{j+l}^{k-1}} z(x)dx = f_{j+l}^{k-1}, \quad l = -\frac{s}{2}, \dots, \frac{s}{2}. \tag{2}$$

The predicted values are then defined by the averages of $z(x)$ on the half-intervals i.e. $\tilde{f}_{2j+\lambda}^k = \frac{1}{h_k}\int_{c_{2j+\lambda}^k} z(x)dx$ with $\lambda = 0$ or $-1$. From standard interpolation results we obtain that:

$$(\mathcal{P}_{k-1}^k f^{k-1})_{2j+\lambda} = \sum_{l=-s/2}^{s/2} \beta_l^\lambda f_{j+l}^{k-1},$$

where the coefficients $\beta_l^\lambda$ are

$$\begin{cases} r = 0 \Rightarrow \beta_0^0 = 1, \\ r = 2 \Rightarrow \beta_0^0 = 1, \quad \beta_1^0 = \frac{1}{8}, \\ r = 4 \Rightarrow \beta_0^0 = 1, \quad \beta_1^0 = \frac{22}{128}, \quad \beta_2^0 = -\frac{3}{128}. \end{cases} \tag{3}$$

By (1) it is easy to prove that $\beta_l^{-1} = -\beta_l^0$ and $\beta_l^\lambda = -\beta_{-l}^\lambda$ with $l > 0$.