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An exact penalty function based on the projection matrix

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A B S T R A C T

This paper proposes an exact penalty function based on the projection matrix concept. The proposed penalty function finds solutions which satisfy the necessary optimality conditions of the original problem. Some theoretical results are presented showing that every regular point provides an absolute minimum to the proposed penalty function if and only if it satisfies the necessary conditions of the original constrained problem. As a general rule, penalty functions may have spurious local minima. An advantage of the proposed penalty function is its ability to identify if an obtained minimum is spurious. The proposed penalty function was applied to solve equality constrained problems from the Hock–Schittkowski Collection. Some solutions were obtained more efficiently using the new penalty function than by using a conventional constrained optimization method.

1. Introduction

The penalty function method is an approach to solve constrained nonlinear programming problems [1,2]. The idea of a penalty function method is to replace the constrained optimization problem by an unconstrained problem such that the solution of the unconstrained problem coincides or at least approximates the solution of the original constrained problem. In particular, when the solution of the penalized problem coincides with the solution of the original problem it is said that the objective function of the unconstrained problem is an exact penalty function [3].

Most of the proposed solutions transforms the constrained problem in an unconstrained problem by adding a penalty term to the original objective function. This penalty term prescribes a high cost when the constraints are violated. The unconstrained problem is solved and the penalty parameters are adjusted until the convergence is attained.

Much research has been done to find penalty functions and methods to solve the resulting unconstrained problems. An example of an exact penalty function was proposed by Zangwill [4]. However, the resulting unconstrained problem is non-differentiable. Another exact penalty function method had been proposed by Morrison [5] and revisited by Meng et al. [6,7]. Luenberger [8] proposed a method combining the Gradient Projection Method with a classical penalty function for problems with equality constraints [2,9]. Luenberger [10] explored a Lagrangian based approach to solve problems with n variables and m constraints leading to an unconstrained problem with m additional variables (m Lagrange multipliers). Bertsekas [11] studied the necessary and sufficient conditions for a penalty method to yield an optimal solution solving only one unconstrained minimization. Fletcher [1] proposed continuous and differentiable multiplier functions for problems with equality constraints. Di Pillo [3] made a theoretical review of properties which ensure that minimum points of the penalty function are solutions of the original constrained problem. Huyer and Neumaier derived continuously differentiable exact penalty functions [12]. They achieved this augmenting the dimension of the problem by a variable that controls both the

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weight of the penalty terms and the regularization of the nonsmooth terms. Rubinov et al. [13] studied single constraint problems and considered nonlinear Lagrange and penalty functions for optimization problems with a single constraint. Antczak [14] proposes an exact penalty function called l_1 exact exponential function and used it to solve nonconvex optimization problems.

Commonly, penalty functions have been used to allow convergence from poor initial points. Fletcher and Leyffer [15] proposed filter methods as an alternative to penalty functions. Filter methods treat constrained nonlinear optimization as a bi-objective optimization problem that minimizes the objective and the constraint violation. Filter methods represent a new way of solving nonlinear constrained problems, though they can also use penalty concepts. Nie [16] proposed the Sequential Penalty Quadratic Programming Filter (SlQP) algorithm based on a modification of the original filter idea. Nie replaced the objective function by a penalized function. The SlQP method does not need Lagrangian multipliers, strong decrease condition and large penalty parameters.

The research in penalty functions has renewed interest because of their application with hybrid metaheuristic methods which usually have difficulties in dealing with constraints and generating feasible solutions [17,18].

This work proposes a new exact penalty function for non-linear programming problems with nonlinear equality constraints based on the projection matrix concept [19,9,20]. The proposed penalty function belongs to the category of exact penalty functions i.e. it allows an exact solution of the original problem without the necessity of using sequential solutions or penalty parameters. The proposed penalty function works in the variables' space of the original constrained problem. Although the concept of the Lagrangian multiplier is implicitly embedded in the proposed penalty function, it is not a Lagrangian approach because it works with the same number of variables of the original problem [21,22].

In sequence, Section 2 reviews some penalty functions, Section 3 presents the new penalty function, Section 4 discusses some theoretical results for the proposed penalty function, Section 5 considers single constraint problems as a theoretical example and reports computational results of the new penalty function applied to solve equality constrained problems from the Hock–Schittkowski Collection [23]. Conclusions and remarks are discussed in Section 6.

2. Penalty functions

Let us consider the nonlinear equality constrained optimization problem,

$$\begin{aligned} &\text{Minimize} \quad f(x),\\ &\text{subject to} \quad h(x) = 0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, f(x) \in \mathbb{R}, h \in \mathbb{R}^m, f \in C^1$ and $h \in C^1$. Assume that the constraints $h(x)$ do not admit irregular points i.e. $\forall x$ such that $h(x) = 0, \nabla h(x)$ is full rank. The regular point consideration is essential for the Karush–Kuhn–Tucker (KKT) necessary conditions to be valid. The KKT conditions are used by almost all optimization methods [2]. For example, with irregular solutions the existence of the Lagrange multipliers cannot be guaranteed and we cannot work with Lagrangian techniques [24,13].

2.1. The classical penalty functions

Classical penalty function methods [2,1] replace the constrained problem by an unconstrained problem of the form,

$$\text{Minimize} \quad f(x) + c_k \Gamma(x), \tag{2}$$

where c_k is a positive constant, $\Gamma \in \mathbb{R}$ is a function satisfying: (i) Γ is continuous, (ii) $\Gamma(x) \geqslant 0, \ \forall x \in \mathbb{R}^n$, and (iii) $\Gamma(x) = 0$ if and only if $h(x) = 0$.

To solve the original constrained problem (1) it is necessary to solve a sequence of unconstrained optimization problems of type (2), with an increasing sequence of c_k. Let $c_k, k = 1, 2, \ldots$ be an unlimited sequence where $c_k \geqslant 0, c_{k+1} > c_k$. With additional hypotheses [2], any limit point of sequence c_k is a solution of the original problem (1).

A classical penalty function is $\Gamma(x) = \frac{1}{2} h(x)^T h(x)$. If the constraints and the objective function are differentiable the unconstrained problem obtained using this penalty function is smooth, but this penalty function is not exact [2].

2.2. Exact penalty function methods

A penalty function $\Gamma(X)$ is said to be exact if there is some c^* finite for which all $c \geqslant c^*$, if x^* is a solution of the unconstrained problem (2) then x^* is also a solution for the original constrained problem (1). When using exact penalty functions, it is not necessary to solve an infinite sequence of penalized problems to obtain the correct solution.

The exact penalty function proposed by Zangwill [4] was

$$\Gamma(x) = \sum_{i=1}^{m} |h_i(x)|. \tag{3}$$

However, the resulting unconstrained problem (2) is non-differentiable.