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Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls

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ABSTRACT

This paper investigates the existence and uniqueness of mild solutions for a class of Sobolev type fractional nonlocal abstract evolution equations in Banach spaces. We use fractional calculus, semigroup theory, a singular version of Gronwall inequality and Leray–Schauder fixed point theorem for the main results. A new kind of Sobolev type appears in terms of two linear operators is introduced. To extend previous works in the field, an existence result of optimal multi-control pairs governed by the presented system is proved. Finally, an example is also given to illustrate the obtained theory.

1. Introduction

We are concerned with the following fractional nonlocal evolution system of Sobolev type

$$^{C}D_t^{\alpha}[LMu(t)] = Eu(t) + f(t, W(t)), \tag{1.1}$$

$$Nu(0) + h(u(t)) = u_0, \tag{1.2}$$

where $^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha, 0 < \alpha \le 1$, and $t \in J = [0, a]$. Let X, Y and Z be three Banach spaces such that Z is densely and continuously embedded in X, the unknown function $u(\cdot)$ takes its values in X and $u_0 \in X$. We assume that the operators $E : D(E) \subset X \to Z, M : D(M) \subset X \to Y, L : D(L) \subset Y \to Z$ and $N : D(N) \subset X \to X, W(t) = (B_1(t)u(t), \ldots, B_r(t)u(t))$ such that $\{B_i(t) : i = 1, \ldots, r, t \in J\}$ is a family of linear closed operators defined on dense sets $S_1, \ldots, S_r$ respectively in X with values in Z. It is also assumed that $f : J \times X^r \to Z$ and $h : C(J : X) \to X$ are given abstract functions satisfying some conditions to be specified later.

The recent science shows that fractional differential equations accurately and perfectly can be applicable in many fields, such as, mathematical modeling of physical, engineering and biological phenomena, and also have motivated several researchers to explore theoretical as well as practical aspects of the subject. Fractional calculus also provides an excellent tool to describe the hereditary properties of various materials and processes. In fact, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetics. More about the development of theory, methods and applications of this matter can be found in [1–5], see also the papers [6–10]. There has been a significant development in nonlocal problems for fractional differential equations or inclusions (see for instance [11–19]).

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On the other hand, there could be no manufacturing, no vehicles, no computers and no regulated environment without control systems. Control systems are most often based on the principle of feedback, whereby the signal to be controlled is compared to a desired reference signal and the discrepancy used to compute corrective control action [20]. The fractional optimal control of a distributed system is an optimal control for which system dynamics are defined with fractional differential equations, see Ozdemir et al. [21]. The existence of optimal pairs of systems governed by fractional evolution equations with initial and nonlocal conditions is also presented by Wang et al. [19] and Wang and Zhou [22].

Moreover, Sobolev type semilinear equations serve as an abstract formulation of partial differential equation which arises in various applications such as in the flow of fluid through fissured rocks, thermodynamics and shear in second order fluids and so on. Moreover, the fractional differential equations of Sobolev type appear in the theory of control of dynamical systems, when the controlled system or/and the controller is described by a fractional differential equation of Sobolev type. Furthermore, the mathematical modeling and simulations of systems and processes are based on the description of their properties in terms of fractional differential equation of Sobolev type. These new models are more adequate than previously used integer order models, so fractional order differential equations of Sobolev type have been investigated by many researchers, see for example, Feckan et al. [23] and Li, Liang and Xu [24].

Motivated by the above facts, we introduce a new kind of Sobolev type for nonlinear fractional evolution equations which are given in terms of two linear operators, this kind requires to formulate two new characteristic solution operators and their properties, such as boundedness and compactness. Further, we present a class of admissible multi-controls and we prove, under an appropriate set of sufficient conditions, an existence result of optimal multi-controls for a Lagrange problem (*LP*).

The paper is organized as follows. In Section 2, we present some essential facts that will be used for the main results, such as, fractional calculus, fractional powers of the generator of an analytic compact semigroup and introduce the form of mild solutions of (1.1)–(1.2). In Section 3, we study the existence and uniqueness of mild solutions of the mentioned system. In Section 4, we prove the existence of optimal pairs for a Lagrange problem. The last section is devoted to giving an example to illustrate the applications of the abstract results.

2. Preliminaries

In this section, we introduce some basic definitions, notations and lemmas, which will be used throughout the work. In particular, we give main properties of fractional calculus [3,4] and well known facts in semigroup theory [25,26].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b],\mathbb{R}^+)$ is given by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the classical gamma function. If $a = 0$, we can write $I^\alpha f(t) = (g_\alpha * f)(t)$, where

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leqslant 0, \end{cases}$$

and as usual, $*$ denotes the convolution of functions. Moreover, $\lim_{\alpha\to 0} g_\alpha(t) = \delta(t)$, with δ the delta Dirac function.

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathbb{N}$, is given by

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,$$

where the function f has absolutely continuous derivatives up to order $(n-1)$.

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathbb{N}$, is given by

$$^C D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0,$$

where the function f has absolutely continuous derivatives up to order $(n-1)$.

Remark 2.1. The following properties hold. Let $n-1 < \alpha < n, n \in \mathbb{N}$

(i) If $f \in C^n([0,\infty))$, then

$$^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0.$$