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An analysis of a new family of eighth-order optimal methods

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ABSTRACT

A new family of eighth order optimal methods is developed and analyzed. Numerical experiments show that our family of methods perform well and in many cases some members are superior to other eighth order optimal methods. It is shown how to choose the parameters to widen the basin of attraction.

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1. Introduction

There are many multistep methods for the solution of nonlinear equations, see e.g. Traub [1], and the recent book by Petković et al. [2]. The idea of optimality in such methods was introduced by Kung and Traub [3] who also developed optimal multistep method of increasing order. For example, the fourth-order optimal method given in [3] is

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)}\right]^2}. \end{cases} \tag{1}$$

Based on this method Chun and Neta [4] constructed and analyzed the sixth order method

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n = w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)}\right]^2}, \\ x_{n+1} = s_n - \frac{f(s_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)} - \frac{f(s_n)}{f(x_n)}\right]^2}. \end{cases} \tag{2}$$

In this paper we will use the idea of weight function to develop a family of optimal eighth order methods and show how to choose the parameters to obtain the best basins of attraction.

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2. An optimal eighth-order method

We consider here a generalization of the Chun–Neta sixth order scheme (2). The new family is constructed using the idea of weight functions. The multistep method is given by

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n = w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{[1-r_n]^2}, \\ x_{n+1} = s_n - \frac{f(s_n)}{f'(x_n)} \frac{1}{[1-H(r_n)J(t_n)P(q_n)]^2}, \end{cases} \tag{3}$$

where $r_n = \frac{f(w_n)}{f(x_n)},\ t_n = \frac{f(s_n)}{f(x_n)},\ q_n = \frac{f(s_n)}{f(w_n)}$ and $H(r),\ J(t),\ P(q)$ are real-valued weight functions to be determined later.

For the method defined by (3), we have the following analysis of convergence.

Theorem 2.1. *Let $\xi \in I$ be a simple zero in an open interval I of a sufficiently differentiable function $f : I \to \mathbb{R}$. Let $e_n = x_n - \xi$. Then the new family of methods defined by (3) is of optimal eighth-order when*

$$\begin{aligned} &H(0)J(0)P(0) = 2, \\ &H'(0)P(0)J(0) = -1, \\ &H''(0)P(0)J(0) = -1, \\ &H'''(0)P(0)J(0) = 3, \\ &|H^{(4)}(0)| < \infty, \\ &J'(0) = -3J(0)/8, \\ &|J''(0)| < \infty, \\ &P'(0) = -P(0)/4, \\ &|P''(0)| < \infty. \end{aligned}$$

The error at the $(n+1)$th step, e_{n+1}, satisfies the relation

$$\begin{aligned} e_{n+1} = c_2 \Bigg[&\left(\frac{2P''(0)}{P'(0)} - \frac{1}{4}\right) c_3^3 - c_2 c_3 c_4 + \left(4 - \frac{3P''(0)}{P'(0)}\right) c_2^2 c_3^2 + 2c_2^3 c_4 + \left(\frac{1}{3}P'(0)J(0)H^{(4)}(0) + \frac{6P''(0)}{P'(0)} - \frac{29}{4}\right) c_2^4 c_3 \\ &+ \left(\frac{1}{2} - \frac{2}{3}P'(0)J(0)H^{(4)}(0) - \frac{4P''(0)}{P'(0)}\right) c_2^6 \Bigg] e_n^8 + O(e_n^9), \end{aligned} \tag{4}$$

where c_i are given by

$$c_i = \frac{f^{(i)}(\xi)}{i! f'(\xi)}, \quad i \geqslant 1. \tag{5}$$

Proof. Let $e_n = x_n - \xi,\ e_n^w = w_n - \xi$ and $e_n^s = s_n - \xi$. Using the Taylor expansion of $f(x)$ around $x = \xi$ and taking $f(\xi) = 0$ into account, we get

$$f(x_n) = f'(\xi)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)\right] \tag{6}$$

and

$$f'(x_n) = f'(\xi)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + O(e_n^7)\right]. \tag{7}$$

Dividing (6) by (7) gives

$$\begin{aligned} u_n &= \frac{f(x_n)}{f'(x_n)} \\ &= e_n - c_2 e_n^2 + (-2c_3 + 2c_2^2)e_n^3 + (-3c_4 + 7c_2c_3 - 4c_2^3)e_n^4 + (10c_2c_4 - 4c_5 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)e_n^5 + (17c_4c_3 \\ &\quad - 28c_4c_2^2 + 13c_2c_5 - 5c_6 - 33c_2c_3^2 + 52c_3c_2^3 - 16c_2^5)e_n^6 + (-92c_3c_2c_4 + 22c_3c_5 - 18c_3^3 + 126c_3^2c_2^2 - 128c_3c_2^4 \\ &\quad + 12c_4^2 + 72c_4c_2^3 - 36c_5c_2^2 - 6c_7 + 16c_2c_6 + 32c_2^6)e_n^7 + (-7c_8 - 118c_5c_2c_3 + 348c_4c_3c_2^2 + 19c_2c_7 - 64c_2c_4^2 \\ &\quad + 31c_4c_5 - 75c_4c_3^2 - 176c_4c_2^4 + 92c_5c_2^3 + 27c_6c_3 - 44c_6c_2^2 + 135c_2c_3^2 - 408c_3^2c_2^3 + 304c_3c_2^5 - 64c_2^7)e_n^8 + O(e_n^9). \end{aligned} \tag{8}$$