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Asymptotic behavior of the positive solutions of an exponential type system of difference equations

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ABSTRACT

In this paper we study the existence of a unique positive equilibrium, the boundedness, persistence and global attractivity of the positive solutions of a system of the following two difference equations:

$$x_{n+1} = ax_n + by_{n-1}e^{-x_n}, \quad y_{n+1} = cy_n + dx_{n-1}e^{-y_n}, \quad n = 0, 1, \ldots$$

where a,b,c,d are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

1. Introduction

In [16] the authors obtained results concerning the global behavior of the positive solutions for the difference equation:

$$x_{n+1} = ax_n + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers. This equation can be considered as a biological model, since it arises from models studying the amount of litter in a perennial grassland.

In addition, in [28] the authors studied analogous results for the system of difference equations:

$$x_{n+1} = ay_n + bx_{n-1}e^{-y_n}, \quad y_{n+1} = cx_n + dy_{n-1}e^{-x_n},$$

where a,b,c,d are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive numbers.

In particular, in this paper we obtain results concerning the behavior of the positive solutions for the following system of difference equations:

$$x_{n+1} = ax_n + by_{n-1}e^{-x_n}, \quad y_{n+1} = cy_n + dx_{n-1}e^{-y_n}, \tag{1.2}$$

where a,b,c,d are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive numbers. More precisely, we study the existence of the unique positive equilibrium of (1.2). In addition, we investigate the boundedness and the persistence of the positive solutions of system (1.2). Furthermore, we study the convergence of the solutions of the zero equilibrium (0,0) of

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(1.2). Finally, we investigate the convergence of the positive solutions of (1.2) to the unique positive equilibrium. We note that if (x_n, y_n) is a solution of (1.2) and $x_{-1} = y_{-1}, x_0 = y_0$, then it is obvious that $x_n = y_n$ and x_n is a solution of (1.1). We also note that, the above system could be considered as a model of two interactive litters of two grass types x and y where the amounts of litter x_n and y_n are affecting to each other. More detailed, each year the grass grows anew from the subterranean roots and at the end of the season dies and fall as litter. Then the total amount of litter of the grass type "x" on the ground at the end of the season in a given year depends to the amount remaining from the previous year plus new litter from previous year's growth of the grass type "y". Then the total amount of litter x_{n+1} at the end of year $n+1$ consists of two components. The first component ax_n represents a fraction of the litter remaining from the n. The second component $by_{n-1}e^{-x_n}$ describes the litter produced in year $n-1$ of the grass type "y". Since the nutrients come only from the decay of the previous year's litter, the production of new litter is inhibited by the litter x_n currently on the ground, but increased by the recycling of the previous year's litter y_{n-1}. Finally an analogous expression holds for the total amount of litter y_{n+1} at the end of year $n+1$. Then it is obvious that (x_n, y_n) satisfies a system of difference equations of the form (1.2).

Since difference equations and systems of difference equations containing exponential terms have many potential applications in Biology, there are many papers dealing with such equations. See, for example, [4,16,22,25,26,28,30–33,37] and the references cited therein. We also note that since difference equations have many applications in applied sciences, there is a quite rich bibliography concerning theory and applications of difference equations (see for example [1–26], [28–37] and the references cited therein).

2. Existence and uniqueness of a positive equilibrium for (1.2)

In this section we study the existence and the uniqueness of the positive equilibrium of (1.2).

Theorem 2.1. *The following statements are true:*

I. *Suppose that*

$$a, b, c, d \in (0,1), \quad \theta = \frac{bd}{(1-a)(1-c)} > 1. \tag{2.1}$$

Then system (1.2) has a unique positive equilibrium $(\bar{x}, \bar{y})$. Moreover

$$\frac{1}{1+\frac{d}{1-c}} \ln(\theta) \leqslant \bar{x} \leqslant \ln(\theta), \quad \frac{1}{1+\frac{b}{1-a}} \ln(\theta) \leqslant \bar{y} \leqslant \ln(\theta). \tag{2.2}$$

II. *Consider that a,b,c,d are positive constants such that:*

$$a, b, c, d \in (0,1), \quad \theta \leqslant 1. \tag{2.3}$$

Then, the zero equilibrium (0,0) is the unique equilibrium of system (1.2).

Proof I. We consider the system of algebraic equations:

$$x = ax + bye^{-x}, \quad y = cy + dxe^{-y},$$

or, equivalently:

$$(1-a)x = bye^{-x}, \quad (1-c)y = dxe^{-y}. \tag{2.4}$$

Then, from (2.4), if $x \neq 0$ and $y \neq 0$, we obtain that:

$$x + y = \ln(\theta). \tag{2.5}$$

Relations (2.4) and (2.5) imply that:

$$x = \frac{\ln(\theta)}{1 + \frac{1-a}{b} e^x}, \quad y = \frac{\ln(\theta)}{1 + \frac{1-c}{d} e^y}.$$

We consider the function:

$$F(x) = x - \frac{\ln(\theta)}{1 + \frac{1-a}{b} e^x}.$$

So, from (2.1) we get that $F(0) < 0$ and $\lim_{x\to\infty} F(x) = \infty$. Then, there exists a $\bar{x} \in (0, \infty)$ such that

$$\bar{x} = \frac{\ln(\theta)}{1 + \frac{1-a}{b} e^{\bar{x}}}. \tag{2.6}$$

Similarly, we can prove that there exists a $\bar{y} \in (0, \infty)$, such that: