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On the particular solution of constant coefficient fractional differential equations

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ABSTRACT

The eigenfunction approach to compute the particular solution of constant coefficient ordinary differential equations is extended to the fractional case. It is shown that the exponentials are also the eigenfunctions of such equations. Solutions corresponding to products of powers and exponentials are presented. The singular case is studied and a matricial algorithm for its implementation is presented.

1. Introduction

The computation of particular solutions of integer order constant coefficient ordinary linear equations was considered in some published papers mainly in [1,3]. In a previous paper [5] we studied the problem and proposed an approach based on the concept of eigenfunction. We showed how to compute the solution when the input is an exponential or the product of a power and an exponential. We studied and solved also the singular cases.

In this paper we are going to enlarge those results to the fractional linear equations case. Normally they are written in the general format [4]

$$\sum_{k=0}^{N} a_k D^{\alpha_k} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k} x(t) \tag{1}$$

with $t \in \mathbb{R}$; the symbol D represents the derivative operator. The parameters α_k and β_k are the derivative orders that we assume to form strictly increasing sequences of positive numbers. In the so-called commensurate case we write $\alpha_k = \beta_k = k\alpha$. In current applications we assume that $\beta_M \leqslant \alpha_N$ for stability reasons. We will assume that

$$x(t) = e^{\beta t} t^K \quad t \in \mathbf{R} \tag{2}$$

preventing the use of the Laplace transform (even two-sided) or the Fourier transform [6] to find the particular solution of (1).

The treatment of the fractional case has similarities but also differences with the integer order due to the peculiarities of the fractional derivatives. Knowing that the matter of derivative definition is not pacific we make a brief introduction to the subject. We will use the Grünwald–Letnikov fractional derivative, since the derivative of an exponential is again an exponential.

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The paper outlines as follows. In Section 2 we compute the eigenfunctions of fractional linear systems and corresponding eigenvalues. The obtained results are generalised to the power.exponential inputs and in Section 3 to the singular cases. For this a matricial formulation is presented. Finally we will present some conclusions.

2. The eigenfunctions of differential equations

2.1. On the fractional derivatives

We are going to generalise to the fractional case the results presented in [5]. Before going ahead, let us look into the current definitions of derivative. The first and most frequently used by mathematicians is the following

$$f'_a(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} \tag{3}$$

The second and mostly used in engineering is

$$f'_c(t) = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h} \tag{4}$$

At last we have the symmetric derivative

$$f'_s(t) = \lim_{h\to 0}\frac{f(t+h/2)-f(t-h/2)}{h} \tag{5}$$

When $f(t) = e^{st}$ the three formulations give the same result $f'(t) = se^{st}$, for any $s \in \mathbb{C}$. Return back to the derivative definitions and assume that $h > 0$. As it can be seen, the first uses the actual value, $f(t)$ and the future value, $f(t+h)$: it is an anti-causal derivative. The second is the reverse: it uses the actual value, $f(t)$ and the past value, $f(t-h)$: it is a causal derivative. The third uses simultaneously a future and a past value: it can be called acausal derivative; we will not use it here [4]. These considerations show the reason why engineers prefer the second: our most important systems are described by causal differential equations. This fact becomes clearer when we go into the fractional derivatives.

The incremental ratio fractional derivative are generalisations of the above derivatives for any real or complex order. We define the forward Grünwald–Letnikov[2] derivative by [4]

$$D_f^\alpha f(t) = \lim_{h\to 0+}\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(t-kh)}{h^\alpha}. \tag{6}$$

This derivative generalises (4). Similarly the generalised version of (3) is the backward Grünwald–Letnikov derivative

$$D_b^\alpha f(t) = e^{-j\pi\alpha}\lim_{h\to 0+}\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(t+kh)}{h^\alpha}. \tag{7}$$

In the following we shall be working in the context of the forward derivative (6).

2.2. The exponentials as eigenfunctions

To get the particular solution of fractional differential equations our first task is to compute the derivatives of the exponential. As shown in [4] we have

$$D_f^\alpha e^{st} = s^\alpha e^{st} \quad \text{if } Re(s) > 0 \tag{8}$$

and

$$D_b^\alpha e^{st} = s^\alpha e^{st} \quad \text{if } Re(s) < 0 \tag{9}$$

This is very important. While with integer orders s can be any complex, in the fractional case this does not happen. The fractional derivatives impose a region of convergence that is tied with the causality. The forward derivative is causal while the backward is anti-causal. With these constraints we are able to compute the particular solutions of Eq. (1).

As before [4,5].

- The convolution between two functions $x(t)$ and $y(t)$ be defined by:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\mathrm{d}\tau \tag{10}$$

[2] The terms forward and backward are used here in agreement to the current use in Signal Processing where t is a time that flows from past to future or the reverse.