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# Convexity preservation of five-point binary subdivision scheme with a parameter

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ABSTRACT

A five-point binary subdivision scheme with a parameter $u$ is presented. The generating polynomial method and the Hölder exponent are used to investigate the uniform convergence and $C^k$ continuity of this subdivision scheme, where $k$ depends on the choice of the parameter $u$. Moreover, the conditions on the initial points are discussed for the given limit curve to be convexity preserving and an example is given to illustrate our conclusion.



## 1. Introduction

Subdivision techniques are being widely used to create smooth curves and surfaces in many fields such as in CAGD, CG and geometric modeling. Besides convergence and smoothness, convexity preserving is also a hot topic in curves design. A lot of papers have been published on this topic during the past decades. Cai [1] gave a convexity preserving algorithm for classic four-point interpolating subdivision scheme [2], and claimed that a convex curve can be constructed by means of the four-point interpolating subdivision scheme for arbitrary convex set of discrete points with no three points collinear. Literature [3] discussed the convergence of four-point subdivision scheme in nonuniform control points and its error estimation. Dyn et al. [4] derived the conditions on the parameter of the classic four-point interpolatory subdivision scheme presented in [2] to preserve convexity. Hassan et al. [5] introduced a four-point ternary interpolatory subdivision scheme, that can generate $C^2$ continuous limit curves. The convexity preserving condition was given in [6] for four-point ternary interpolatory subdivision scheme in [5]. Based on quintic B-spline basis functions, Siddiai et al. [7] constructed a binary six-point approximating subdivision scheme, which generates $C^6$ continuous limit curves. The sufficient conditions on the parameters of the scheme to preserve convexity and generate $C^r$ ($r = 7, 8, 9$) limit curves in [7] were introduced by Hao et al. [8]. Under the nonuniform initial control vertices, Kuijit et al. [9] proposed a class of shape preserving binary subdivision schemes, and a sufficient condition for preservation of convexity was deduced. Amat et al. [10] introduced a new approach to prove the convexity preserving properties for interpolatory subdivision schemes through reconstruction operators. A new $C^5$ continuous binary five-point relaxation subdivision scheme with tension parameter was presented by Cao et al. [11].

Based on the above research, we present a five-point binary subdivision scheme with parameter $u$ in this paper and discuss the continuity of the limit curve as well as its convexity preservation. The rest of the paper is organized as follows. In Section 2, a five-point binary subdivision scheme with parameter $u$ is introduced, the continuity of the limit curve is discussed and the Hölder exponent is calculated. In Sections 3 and 4, the convexity preserving properties of limit curves with different choices of parameter are analyzed. Numerical examples and the conclusion is drawn in Section 5.

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## 2. A five point binary subdivision scheme with a parameter

Based on the literature [11], first of all, we give the definition of a new 5-point binary subdivision scheme.

**Definition 1.** Given a set of initial control points $P^0 = \{p_i^0 \in \mathbb{R}^d\}_{i=-2}^{n+2}$, let $P^k = \{p_i^k \in \mathbb{R}^d\}_{i=-2}^{2^k n+2}$ be the set of control points at $k$th level of refinement. Then the refined polygon $\{p_i^{k+1} \in \mathbb{R}^d\}_{i=-2}^{2^{k+1}n+2}$ is obtained by applying the following subdivision rules:

$$\begin{cases} p_{2i}^{k+1} = & \frac{u}{128}p_{i-2}^k + \left(\frac{u}{32}+\frac{3}{16}\right)p_{i-1}^k + \left(\frac{5}{8}-\frac{5u}{64}\right)p_i^k + \left(\frac{u}{32}+\frac{3}{16}\right)p_{i+1}^k + \frac{u}{128}p_{i+2}^k, \\ p_{2i+1}^{k+1} = & \left(\frac{1}{32}+\frac{u}{32}\right)p_{i-1}^k + \left(\frac{15}{32}-\frac{u}{32}\right)p_i^k + \left(\frac{15}{32}-\frac{u}{32}\right)p_{i+1}^k + \left(\frac{1}{32}+\frac{u}{32}\right)p_{i+2}^k. \end{cases} \tag{1}$$

In practice, the new five-point binary subdivision scheme is the special case of literature [11] when $u = 128w$.

The convergence and up to $C^5$ continuity of the five-point binary subdivision scheme (1) are analyzed and discussed as follows.

**Theorem 1.** *The new five-point binary subdivision scheme (1) generates a limiting curve of continuity up to $C^5$.*

**Proof.** From subdivision rules it follows:

$$a_{-4} = a_4 = \frac{u}{128}, \quad a_{-3} = a_3 = 0, \quad a_{-2} = a_2 = \frac{u}{32} + \frac{3}{16}, \quad a_{-1} = a_1 = \frac{15}{32} - \frac{u}{32}, \quad a_0 = \frac{5}{8} - \frac{5u}{64}.$$

Thus, the generating polynomial $a(z)$ for the mask of the subdivision scheme can be written as

$$a(z) = \left(\frac{1+z}{2\sqrt{z}}\right)^6 \left(\frac{u}{z} + (4-2u) + uz\right)\Big/2.$$

Then according to literature [12], we have the following generating polynomials for $S_j (j = 1,2,3,4,5,6)$:

$$a^{(1)}(z) = \frac{2z}{1+z}a(z) = \left(\frac{1+z}{2\sqrt{z}}\right)^5 \left(u + (4-2u)z + uz^2\right)/2\sqrt{z},$$

$$a^{(2)}(z) = \left(\frac{2z}{1+z}\right)^2 a(z) = \left(\frac{1+z}{2\sqrt{z}}\right)^4 \left(u + (4-2u)z + uz^2\right)/2,$$

$$a^{(3)}(z) = \left(\frac{2z}{1+z}\right)^3 a(z) = \left(\frac{1+z}{2\sqrt{z}}\right)^3 \left(uz + (4-2u)z^2 + uz^3\right)/2\sqrt{z},$$

$$a^{(4)}(z) = \left(\frac{2z}{1+z}\right)^4 a(z) = \left(\frac{1+z}{2\sqrt{z}}\right)^2 \left(uz + (4-2u)z^2 + uz^3\right)/2,$$

$$a^{(5)}(z) = \left(\frac{2z}{1+z}\right)^5 a(z) = \left(\frac{1+z}{2}\right) \left(uz + (4-2u)z^2 + uz^3\right)/2,$$

$$a^{(6)}(z) = \left(\frac{2z}{1+z}\right)^6 a(z) = \frac{1}{2}\left(uz^2 + (4-2u)z^3 + uz^4\right).$$

Therefore,

$$\sum_{i\in\mathbb{Z}} a_{2i}^{(j)} = \sum_{i\in\mathbb{Z}} a_{2i+1}^{(j)} = 1, \quad j = 0,1,2,3,4,5,$$

and when $-9 < u < 14.4$, we have

$$\left\|\frac{1}{2}S_1\right\|_\infty = \max\left\{\left|\frac{3u}{128}+\frac{1}{32}\right| + \left|\frac{5}{16}-\frac{5u}{128}\right| + \left|\frac{u}{128}+\frac{5}{32}\right| + \left|\frac{u}{128}\right|\right\} < 1;$$

when $-6 < u < 10$, we have

$$\left\|\frac{1}{2}S_2\right\|_\infty = 2\max\left\{\left|\frac{u}{64}\right| + 2\left|-\frac{u}{128}+\frac{1}{8}\right|,\ 2\left|\frac{u}{64}+\frac{1}{32}\right| + \left|-\frac{u}{32}+\frac{3}{16}\right|\right\} < 1,$$

$$\left\|\frac{1}{2}S_3\right\|_\infty = 4\max\left\{\left|\frac{u}{128}\right| + \left|\frac{u}{128}+\frac{1}{32}\right| + \left|-\frac{u}{64}+\frac{3}{32}\right|\right\} < 1;$$

when $-2 < u < 6$, we have