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Nonautonomous impulsive systems with unbounded nonlinear terms ☆

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ABSTRACT

In Fenner and Pinto (1999) [23], proved that if the linear impulsive system satisfies the IS condition (see Definition 2.1) with bounded nonlinear term $f(t,x,\eta)$, then the perturbed nonlinear impulsive system has a unique bounded solution (see Theorem A). The method used to prove Theorem A cannot be applied to the unbounded case. In this paper, we prove that if $|f(t,x,\eta)| \leqslant \mu e^{\beta|t|} + M$, then the perturbed nonlinear impulsive system has a unique solution.

1. Introduction

The theory of IDEs arises in realistic models in biology, physics, technology, neural networks, see for example [1–10] and the references cited therein. On the other hand, the theory of IDEs has many applications in real world. For examples, impulsive systems are at the heart of many chaos control techniques as well as algorithms for chaotification of otherwise periodic systems (see [11–13]). Another important application is impulsive systems in models that give rise to power laws and fat tailed distributions. Here positive impulses accumulate to give rise to a rich-get-richer phenomenon that is known as the Matthew effect (see [14]). A systematic study on impulsive differential equations was presented in the monographs (Stamova [15], Lakshmikantham et al. [16], Bainov and Simeonov[17], Bainov and Kostadinov [18], Samoilenko and Perestyuk [19], Fu et al. [20]). One of basic theories to study the IDEs is the existence and uniqueness of the solution to the IDEs. Adopting dichotomy theory (e.g. [21,22]), Fenner and Pinto [23] proved that if the linear impulsive system satisfies the IS condition (see Definition 2.1) with bounded nonlinear term $f(t,x,\eta)$, then the perturbed nonlinear impulsive system has a unique bounded solution (see Theorem A in next section). In this paper, we prove that if $|f(t,x,\eta)| \leqslant \mu e^{\beta|t|} + M$, then the perturbed nonlinear impulsive system has a unique solution $x(t)$ satisfying $|x(t)| = O(e^{\beta|t|})$.

We note that the method used in this paper is motivated partly by [24]. However, the problem considered in this paper is completely different from that in [24]. In [24] the authors assume that the nonlinear term $f(t,x)$ is bounded, and also the authors consider continuous ODEs without impulses.

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2. Main results

In this section we state our main results. First, we introduce some definitions. Let W be a Banach space. A linear nonautonomous system with impulses at times $\{t_k\}_{k\in\mathbb{Z}}$ is described by

$$\begin{cases} \dot{x}(t) &= A(t)x(t), \quad t \neq t_k, \\ \Delta x(t_k) &= \tilde{A}(t_k)x(t_k), \quad k \in \mathbb{Z}, \end{cases} \tag{2.1}$$

A perturbed nonautonomous system with impulses is described by

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t,x,\eta), \quad t \neq t_k, \\ \Delta x(t_k) &= \tilde{A}(t_k)x(t_k) + \tilde{f}(t_k,x,\eta), \quad k \in \mathbb{Z}, \end{cases} \tag{2.2}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-),\ x(t_k^-) = x(t_k)$ represents the jump of the solution $x(t)$ at $t = t_k,\ \eta \in W,\ f,\ \tilde{f} : \mathbb{R}\times\mathbb{R}^n\times W \to \mathbb{R}^n$ are continuous functions. A fundamental matrix of system (2.1) is given by

$$\mathbf{X}(t) = \Phi(t)\prod_{t_k\in[t_0,t)}\Phi^{-1}(t_k)(I+\tilde{A}(t_k))\Phi(t_k)\Phi^{-1}(t_0), \quad t \geqslant t_0,$$

where $\Phi(t)$ is any fundamental matrix of the differential system $\dot{x}(t) = A(t)x(t)$, provided that $\Phi(t_k)$ is invertible, for all $t_k \geqslant t_0$. In what follows, we shall assume that $\mathbf{X}(t)$ is invertible for all $t \in \mathbb{R}$.

Definition 2.1 [23]. The impulsive system (2.1) is said to satisfy an int-summable condition (IS condition), if there exists a projection P and $\kappa \geqslant 0$ such that

$$\int_{-\infty}^{t}|\mathbf{X}(t)P\mathbf{X}^{-1}(s)|ds + \int_{t}^{+\infty}|\mathbf{X}(t)(I-P)\mathbf{X}^{-1}(s)|ds + \sum_{t_k\in(-\infty,t)}|\mathbf{X}(t)P\mathbf{X}^{-1}(t_k^+)| + \sum_{t_k\in[t,+\infty)}|\mathbf{X}(t)(I-P)\mathbf{X}^{-1}(t_k^+)| \leqslant \kappa$$

uniformly in $\mathbb{R}$.

Definition 2.2. The impulsive system (2.1) is said to possess an exponential dichotomy, if there exists a projection P and constants $K > 0,\ \alpha > 0$ such that

$$\begin{cases} |\mathbf{X}(t)P\mathbf{X}^{-1}(s)| \leqslant Ke^{-\alpha(t-s)}, \text{ for } \quad t \geqslant s,\ t,\ s \in \mathbb{R}, \\ |\mathbf{X}(t)(I-P)\mathbf{X}^{-1}(s)| \leqslant Ke^{\alpha(t-s)}, \text{ for } t \leqslant s,\ t,\ s \in \mathbb{R}. \end{cases} \tag{2.3}$$

In [23], Fenner and Pinto proved the following result:

Theorem A. *Suppose the linear impulsive system (2.1) satisfies the IS condition. If the nonlinear terms $f(t,x,\eta),\ \tilde{f}(t,x,\eta)$ satisfy*

$(\tilde{H}_1)$ $|f(t,x,\eta)| \leqslant \mu,\ |\tilde{f}(t,x,\eta)| \leqslant \mu$;
$(\tilde{H}_2)$ $|f(t,x_1,\eta) - f(t,x_2,\eta)| \leqslant r|x_1 - x_2|$;
$|\tilde{f}(t,x_1,\eta) - \tilde{f}(t,x_2,\eta)| \leqslant r|x_1 - x_2|$;
$(\tilde{H}_3)$ $\kappa r < 1$,
then system (2.2) has a unique bounded solution.

To obtain Theorem A, one of the essential conditions is $(\tilde{H}_1)$. Our main results show what happens when the nonlinear term is unbounded and $|f(t,x,\eta)| \leqslant \mu e^{\beta|t|} + M$.

Theorem 2.1. *Suppose the linear impulsive system (2.1) admits an exponential dichotomy with the estimates (2.3). For $t \in \mathbb{R},\ x,\ x_1,\ x_2 \in \mathbb{R}^n,\ \eta \in W$, if $f(t,x,\eta),\ \tilde{f}(t,x,\eta)$ satisfy*

(H_1) $|f(t,x,\eta)| \leqslant \mu e^{\beta|t|} + M,\ |\tilde{f}(t,x,\eta)| \leqslant \mu e^{\beta|t|} + M$;
(H_2) $|f(t,x_1,\eta) - f(t,x_2,\eta)| \leqslant r|x_1 - x_2|$;
$|\tilde{f}(t,x_1,\eta) - \tilde{f}(t,x_2,\eta)| \leqslant r|x_1 - x_2|$;
(H_3) $2Kr(\alpha-\beta)^{-1} + 2KrN\left(1 + \frac{1}{1-e^{-(\alpha-\beta)}}\right) < \frac{1}{2}$,

where $\mu,\ r,\ \beta,\ M$ and $\alpha - \beta$ are all positive constants, N which denotes that the interval $[n, n+1)$ contains no more than N terms of the sequences $\{t_k\}_{k\in\mathbb{Z}}$, then system (2.2) has a unique solution $x(t)$ satisfying

$$|x(t)| = O(e^{\beta|t|}). \tag{2.4}$$