Applied Mathematics and Computation 245 (2014) 438–446

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Infinitely many nontrivial periodic solutions for damped vibration problems with asymptotically linear terms ☆

Guanwei Chen

School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, Henan Province, China

ARTICLE INFO

Keywords:
Damped vibration problems
Infinitely many nontrivial periodic solutions
Asymptotically quadratic
Variational method

ABSTRACT

In this paper, we study a class of damped vibration problem with asymptotically quadratic terms at infinity. We obtain infinitely many nontrivial periodic solutions by variational method. To the best of our knowledge, there is no published result focusing on this class of damped vibration problem with asymptotically quadratic terms at infinity.

1. Introduction and main results

We shall study the existence of infinitely many nontrivial periodic solutions for the following damped vibration problem

$$\begin{cases} \ddot{u} + (q(t)I_{N\times N} + B)\dot{u} + \left(\frac{1}{2}Bq(t) - A(t)\right)u + H_u(t,u) = 0, & t \in \mathbb{R}, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, & T > 0, \end{cases} \tag{1.1}$$

where $u = u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$, $I_{N\times N}$ is the $N \times N$ identity matrix, $q(t) \in L^1(\mathbb{R}; \mathbb{R})$ is T-periodic and satisfies $\int_0^T q(t)dt = 0$, $A(t) = [a_{ij}(t)]$ is a T-periodic symmetric $N \times N$ matrix-valued function with $a_{ij} \in L^\infty(\mathbb{R}; \mathbb{R})$ $(\forall i, j = 1, 2, \ldots, N)$, $B = [b_{ij}]$ is an antisymmetric $N \times N$ constant matrix, $H(t,u) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t and $H_u(t,u)$ denotes its gradient with respect to the u variable.

If $B = A(t) = 0$ (zero matrices) and $q(t) \equiv 0$, the non-autonomous second order systems (1.1) have been deeply studied and a lot of existence and multiplicity results obtained, for example, see [1–6]. When $B = 0$ (zero matrix), the authors [7] have studied (1.1) and obtained the existence and multiplicity of periodic solutions. Recently, the authors [8] have studied a more general problem (that is, damped vibration problem (1.1) with $B \neq 0$) by using a critical point theorem in [9] and a symmetric mountain pass theorem in [10]. The authors [8] have obtained two multiplicity results of nontrivial periodic solutions of (1.1) if $H(t,u)$ is even in u, and $H(t,u)$ satisfying

$$\limsup_{|u|\to 0} \frac{H(t,u)}{|u|^2} \leqslant 0 \text{ uniformly in } t \in [0,T] \tag{1.2}$$

and following local Ambrosetti–Rabinowitz **superquadratic** condition:

$$0 < \mu H(t,u) \leqslant (H_u(t,u), u), \quad \forall t \in [0,T], \ \forall |u| \geqslant r, \tag{1.3}$$

where $\mu > 2$ and $r \geqslant 0$ are some constants, $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^N$ and the associated norm is denoted by $|\cdot|$.

☆ Research supported by the Tianyuan Fund for Mathematics of NSFC (Grant No. 11326113) and the Key Project of Natural Science Foundation of Educational Committee of Henan Province of China (Grant No. 13A110015).

E-mail address: guanweic@163.com

http://dx.doi.org/10.1016/j.amc.2014.07.114
0096-3003/

As usual, we say H satisfies the *asymptotically quadratic* (or *superquadratic*) growth condition at infinity if

$$\lim_{|u|\to\infty}\frac{|H(t,u)|}{|u|^2}=V(t)\quad\left(\text{or}\ \lim_{|u|\to\infty}\frac{|H(t,u)|}{|u|^2}=+\infty\right)\quad\text{uniformly in } t\in[0,T],$$

where $V(t)$ satisfies $\inf_{t\in[0,T]}V(t)>0$. Obviously, condition (1.3) implies $\lim_{|u|\to\infty}\frac{|H(t,u)|}{|u|^2}=+\infty$. Inspired by Li et al. [8] (**superquadratic** case at infinity), we shall study (1.1) with $H(t,u)$ being **asymptotically quadratic** growth at infinity.

To state our main result, we assume that

(H_1) There are constants $\mu\in(1,2)$ and $c_1,c_2,c_3>0$ such that

$$c_3|u|^{\mu}\leqslant H(t,u)\leqslant c_1|u|,\quad \forall t\in[0,T]\ \text{and}\ |u|\leqslant c_2.$$

(H_2) H(t,u)$\geqslant\frac{1}{2}(H_u(t,u),u)\geqslant 0,\ \forall(t,u)\in[0,T]\times\mathbb{R}^N$.

(H_3) $\lim_{|u|\to\infty}\frac{H(t,u)}{|u|^2}=V(t)$ uniformly in $t\in[0,T]$, where $\inf_{t\in[0,T]}V(t)>0$.

(H_4) $\widetilde{H}(t,u)\to+\infty$ as $|u|\to\infty$ and

$$\limsup_{|u|\to 0}\frac{|H_u(t,u)|^{\frac{\mu}{\mu-1}}}{\widetilde{H}(t,u)}=P(t)\ \text{uniformly in}\ t\in[0,T],$$

where $\widetilde{H}(t,u):=H(t,u)-\frac{1}{2}(H_u(t,u),u)$ and $|P(t)|<\infty$.

Remark 1.1. Compared with the condition (1.2) used in [8], our condition (H_1) implies $\lim_{|u|\to 0}\frac{H(t,u)}{|u|^2}=+\infty$, i.e., H is *subquadratic at 0*. Besides, condition (1.3) used in [8] implies H is *superquadratic at infinity*, but our condition (H_3) implies H is *asymptotically quadratic at infinity*. To the best of our knowledge, there is no published result focusing on the existence (or multiplicity) of nontrivial periodic solutions for (1.1) with H being *asymptotically quadratic at infinity* and *subquadratic at 0*.

Now, our main result reads as follows:

Theorem 1.1. *If (H_1)–(H_4) hold and $H(t,u)$ is even in u, then (1.1) possesses infinitely many nontrivial T-periodic solutions.*

Notations. In this paper, we shall use $\|\cdot\|_p$ and $(\cdot,\cdot)_p$ to denote the norm and the corresponding inner product of $L^p([0,T];\mathbb{R}^N)$ for $\forall p\in[1,\infty]$, respectively. Besides, we use $(\cdot,\cdot)$ and $|\cdot|$ to denote the standard inner product and the associated norm in $\mathbb{R}^N$, respectively.

The rest of our paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1), and give some preliminary lemmas, which are useful in the proof of our result, and then we give the detailed proof of our main result.

2. Variational frameworks and the proof of our main result

In this section, we shall assume that (H_1)–(H_4) hold and $H(t,u)$ is even in u.

Let $W:=H_T^1$ be defined by

$$H_T^1:=\left\{u=u(t):[0,T]\to\mathbb{R}^N|\ u\ \text{is absolutely continuous},\ u(0)=u(T),\ \text{and}\ \dot{u}\in L^2([0,T];\mathbb{R}^N)\right\}$$

with the inner product

$$(u,v)_W:=\int_0^T[(u,v)+(\dot{u},\dot{v})]dt,\quad \forall u,v\in W,$$

where $(\cdot,\cdot)$ has been defined in Notations. The corresponding norm of $(\cdot,\cdot)_W$ is defined by $\|u\|_W=(u,u)_W^{1/2}$. Obviously, W is a Hilbert space.

Let

$$Q(t):=\int_0^t q(s)ds$$

and

$$\|u\|_0:=\left(\int_0^T e^{Q(t)}\left(|u|^2+|\dot{u}|^2\right)dt\right)^{1/2},\quad u\in W,$$

where the function q is defined in problem (1.1). We denote by $\langle\cdot,\cdot\rangle_0$ the inner product corresponding to $\|\cdot\|_0$ on W. Obviously, the norm $\|\cdot\|_0$ is equivalent to the usual one $\|\cdot\|_W$ on W.