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Nonlinear analysis in a modified van der Pol oscillator

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ABSTRACT

In this paper we study the nonlinear dynamics of a modified van der Pol oscillator. More precisely, we study the local codimension one, two and three bifurcations which occur in the four parameter family of differential equations that models an extension of the classical van der Pol circuit with cubic nonlinearity. Aiming to contribute to the understand of the complex dynamics of this system we present analytical and numerical studies of its local bifurcations and give the corresponding bifurcation diagrams. A complete description of the regions in the parameter space for which multiple small periodic solutions arise through the Hopf bifurcations at the equilibria is given.

1. Introduction and statement of the main results

In this paper we study the local codimension one, two and three bifurcations and the respective qualitative changes in the dynamics of the following system of nonlinear equations

$$\begin{cases} \dot{x} = \frac{dx}{d\tau} = -\nu\left(g(x) + \frac{x-z}{R} + y\right), \\ \dot{y} = \frac{dy}{d\tau} = x - a, \\ \dot{z} = \frac{dz}{d\tau} = \frac{x-z}{R}, \end{cases} \tag{1}$$

where $g(x) = \alpha(x^3/3 - x)$, $(x, y, z) \in \mathbb{R}^3$ are the state variables and the real parameters ν, α, R and a belong to the set

$$\mathcal{T} = \left\{(\nu, \alpha, R, a) \in \mathbb{R}^4 : \nu > 0, \alpha > 0, R > 0, a \geqslant 0\right\}. \tag{2}$$

As far as we know, system (1) was proposed and firstly studied in [2], and it can be obtained from the system

$$\begin{cases} C_0 x' = -\left(f(x) + \frac{x-z}{R} + y\right), \\ L y' = x - a, \\ C z' = \frac{x-z}{R}, \end{cases} \tag{3}$$

where $f(x) = -a_1 x + a_3 x^3$, $a_1 > 0, a_3 > 0$, by the following changes in variables, parameters and rescaling in time

$$x = V_0 \bar{x}, \quad y = \frac{V_0}{\omega L}\bar{y}, \quad z = V_0 \bar{z}, \quad \tau = \omega t, \quad \omega = \frac{1}{\sqrt{LC}}, \quad V_0 = \sqrt{\frac{a_1}{3a_3}},$$

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$$\nu = \frac{C}{C_0}, \quad \bar{a} = \frac{a}{V_0}, \quad \alpha = \frac{a_1}{\omega C}, \quad \bar{R} = R\omega C$$

and then dropping the bars. The prime denotes derivatives with respect to the independent variable t. In Eqs. (3) x and z are the voltage models across the capacitors with capacitances C_0 and C, respectively and y is the current model through the inductor with inductance L. See Fig. 1. In this figure, $f(x)$ represents the nonlinear characteristic of a negative conductance, R is the resistance of the resistor and a is a battery voltage model. See [2] for more details. See also [1] for a study in a similar electronic circuit.

Despite the simplicity of the electronic circuit scheme shown in Fig. 1, the related system (1) has a rich dynamical behavior such that canard orbits according to [2]. System (1) has only one equilibrium point $E_a = (a, g(a), a)$, which exists for any parameter values in $\mathcal{T}$.

In this paper by using the projection method which allows us the calculation of the Lyapunov coefficients associated to the Hopf bifurcations we study all the possible bifurcations (generic and degenerate ones) which occur at the equilibrium E_a of system (1). In this way the analyzes presented in [2] are extended and completed here. Hopf bifurcations give the simplest way in which the solutions of (1) present a periodic oscillatory regime with the birth of one or more limit cycles. More precisely, we prove the following statements:

(a) For the equilibrium $E_0 = (0,0,0)$, that is $a = 0$ (no battery) in system (1), the Hopf surface $\mathcal{H}_1$ is obtained in the space of parameters $(\nu, \alpha, R, 0) \in \mathcal{T}$ and the first Lyapunov coefficient l_1 is calculated. It is established that this coefficient is always negative on the surface $\mathcal{H}_1$.

(b) For the equilibrium $E_a, 0 < a < 1$, the Hopf hypersurface $\mathcal{H}_2$ is obtained in the space of parameters $\mathcal{T}$, the first Lyapunov coefficient l_1 is calculated and it is shown that this coefficient vanishes on a 2-dimensional surface contained in $\mathcal{H}_2$, giving rise to codimension two bifurcations. The second Lyapunov coefficient l_2 is calculated and it is established that this coefficient also vanishes on a 2-dimensional surface contained in $\mathcal{H}_2$, giving rise to codimension three bifurcations along a curve $\mathcal{C}$ given by the intersection of the surfaces $\{l_1 = 0\}$ and $\{l_2 = 0\}$. The third Lyapunov coefficients are obtained for points on the curve $\mathcal{C}$. It is proved that the third Lyapunov coefficient is always negative on $\mathcal{C}$.

From statement (a) it follows that the maximum number of small periodic orbits bifurcating from the equilibrium E_0 is one. See the corresponding bifurcation diagram in Fig. 2. Furthermore, from statement (b) we can deduce that there is a region in the parameter space for which one repelling periodic orbit and two attracting periodic orbits coexist with the repelling equilibrium point E_a, for $0 < a < 1$. See the bifurcation diagrams in Figs. 3,4, 6–9.

The article is organized as follows. In Section 2 through a linear analysis of system (1) we obtain the Hopf surface for E_0 and the Hopf hypersurface for $E_a, 0 < a < 1$. In Section 3 following [3,5,6] we present a brief review of the methods used to study codimension one, two and three Hopf bifurcations, describing in particular how to calculate the Lyapunov coefficients related to the stability of the equilibrium point as well as of the periodic orbits which appear in these bifurcations. In general the Lyapunov coefficients are very difficult to be obtained analytically. These methods are used in Section 4 to prove the main results of this paper, described in statements (a) and (b) above. Finally, in Section 5 we make some concluding remarks.

2. Linear analysis of system (1)

In this section we study some generalities and linear stability of system (1). In a vectorial notation which will be useful in the calculations, system (1) can be written as $\mathbf{x}' = f(\mathbf{x}, \zeta)$, where

$$f(\mathbf{x}, \zeta) = \left(-\nu\left(\alpha\left(\frac{x^3}{3} - x\right) + \frac{x - z}{R} + y\right), x - a, \frac{x - z}{R}\right), \tag{4}$$

$\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and $\zeta = (\nu, \alpha, R, a) \in \mathcal{T}$.

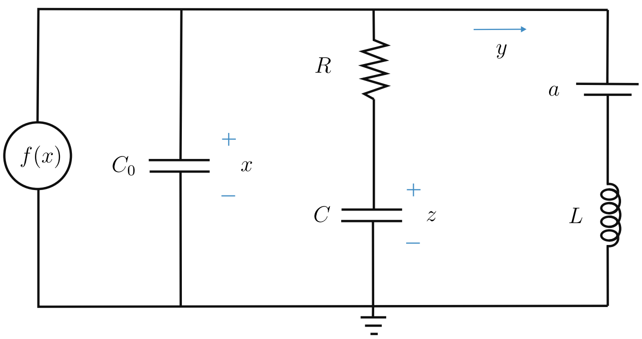

Fig. 1. Circuit scheme.