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A class of rotational solutions for the N-dimensional incompressible Navier–Stokes equations

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ABSTRACT

In this paper, we present a sufficient and necessary condition for the existence of a class of rotational exact solutions $\boldsymbol{u} = \boldsymbol{b}(t) + A(t)\boldsymbol{x}$ for general N-dimensional incompressible Navier–Stokes equations. Such solutions are global and can be explicitly expressed by appropriate formulae. Once the required matrix $A(t)$ is chosen, the solution $\boldsymbol{u}$ is directly obtained.

1. Introduction

The general N-dimensional Navier–Stokes equations can be formulated as follows

$$\operatorname{div}(\boldsymbol{u}) = 0, \tag{1.1}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \mu \Delta \boldsymbol{u}, \tag{1.2}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)^T \in R^N$, $\boldsymbol{u} = (u_1, u_2, \ldots, u_N)^T$ are the components of the N-dimensional velocity field, and $p(\boldsymbol{x}, t)$ the pressure of the fluid at a position $\boldsymbol{x}$ and μ is a non-negative constant.

The Navier–Stokes equations (1.1) and (1.2) are fundamental governing equations for fluid mechanics. The solutions represent fundamental fluid dynamic flows. Owing to the uniform validity of exact solutions, the basic phenomena described by Navier–Stokes equations be more closely studied. The exact solutions also may well be used to test and improve numerical codes for computing more complicated flows. This inherently nonlinear set of partial differential equations has no general solutions, and only a small number of exact solutions have been found. A similarity solution is found which describes the flow impinging on a flat wall at an arbitrary angle of incidence [1]. Wang proved the existing exact solutions of the generalized Beltrami flows, and several new solutions are presented [2]. For unsteady flow, new classes of exact analytical solutions are given representing perturbation over a uniform stream [3]. Zelik's work gives details of the existence of weak solutions for the unbounded domain [4,5]. As $\mu = 0$, the Navier–Stokes equations (1.1) and (1.2) become Euler equations. In 1965, Arnold first introduced the famous Arnold–Beltrami–Childress flow. The solutions exhibit interesting behavior with locally infinite energy [6]. Makino obtained the radial solutions to the Euler and Navier–Stokes equations in 1993, using the separation method [7]. In 1995, Zhang and Zheng obtained spiral solutions for the 2D compressible Euler equations [8]. In 2011, Yuen further obtained a class of exact and rotational solutions for incompressible 3D Euler equations [9,10]. The purpose of this paper is to find a sufficient and necessary condition for the existence of the following vector solutions

$$\boldsymbol{u} = \boldsymbol{b}(t) + A(t)\boldsymbol{x},$$

where N-dimensional vector function $\boldsymbol{b}(t)$ and $N \times N$ matrix function $A(t)$ are defined by

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$$\boldsymbol{b}(t) = (b_1(t), b_2(t), \ldots, b_N(t))^T, \quad A(t) = (a_{ij}(t))_{N\times N}, \tag{1.3}$$

and elements $b_i(t)$ and $a_{ij}(t)$ $(i,j = 1,2,\ldots,N)$ are functions about t. Based on algebraic technique on vectors, matrices and curve integration, we theoretically show the existence of the vector solutions (1.3) for the general N-dimensional incompressible Navier–Stokes equations. Such solutions are global and can be explicitly expressed by appropriate formulae, from which we find a series of exact solutions previously unknown. Special cases of our results include an interesting conclusion: If the velocity field $\boldsymbol{u}$ is a linear transformation on $\boldsymbol{x} \in R^N$, then the pressure p is a relevant quadratic form. This paper is arranged as follows. In Section 2, we show that the N-dimensional incompressible Navier–Stokes equations admit the Cartesian solutions if and only if A is a symmetric matrix with zero-trace. Moreover, some illustrative examples are provided. The structure and properties of such solutions are further analyzed. In Section 3, we give some conclusions and remarks.

2. The incompressible Navier–Stokes equations

2.1. Existence of cartesian solutions

Our main results on the existence of solutions are stated as follows.

Theorem 1. *If the matrices $A = A(t)$ and $B = (A_t + A^2)/2$ satisfy*

$$tr(A) = 0, \quad B^T = B, \tag{2.1}$$

then the N-dimensional incompressible Navier–Stokes equations (1.1) and (1.2) admit explicit exact solutions in the form

$$\boldsymbol{u} = \boldsymbol{b}(t) + A\boldsymbol{x}, \tag{2.2}$$

$$p(\boldsymbol{x}) = -\boldsymbol{b}_t^T\boldsymbol{x} - \boldsymbol{x}^T A\boldsymbol{b} - \boldsymbol{x}^T B\boldsymbol{x} + c(t), \tag{2.3}$$

where $c(t)$ is an arbitrary function of time variable t; the inner product between vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)^T$ and vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_N)^T$ is defined by

$$\boldsymbol{x}^T\boldsymbol{y} = x_1y_2 + x_2y_2 + \cdots + x_Ny_N;$$

and the trace of matrix A is defined by

$$tr(A) = a_{11} + a_{22} + \cdots + a_{NN}.$$

Proof. We only need to verify that the functions (2.2) and (2.3) satisfy the Navier–Stokes system (1.1) and (1.2) under conditions (2.1). Substituting (2.2) into (1.1) gives

$$\mathrm{div}\boldsymbol{u} = \mathrm{div}(\boldsymbol{b} + A\boldsymbol{x}) = \mathrm{div}(\boldsymbol{b}) + \mathrm{div}(A\boldsymbol{x}) = \mathrm{div}(A\boldsymbol{x}) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_{i1}x_1 + \cdots + a_{iN}x_N) = \sum_{i=1}^{N} a_{ii} = \mathrm{tr}(A) = 0. \tag{2.4}$$

We prove how to get the solution (2.3) through solving the Eq. (1.2). Substituting (2.2) into (1.2) produces

$$\begin{aligned}\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \mu\Delta\boldsymbol{u} &= \boldsymbol{b}_t + A_t\boldsymbol{x} + [(\boldsymbol{b} + A\boldsymbol{x})\cdot\nabla](\boldsymbol{b} + A\boldsymbol{x}) + \nabla p, = \boldsymbol{b}_t + A_t\boldsymbol{x} + (\boldsymbol{b}\cdot\nabla)A\boldsymbol{x} + (A\boldsymbol{x}\cdot\nabla)A\boldsymbol{x} + \nabla p \\ &= \boldsymbol{b}_t + A\boldsymbol{b} + (A_t + A^2)\boldsymbol{x} + \nabla p = 0.\end{aligned} \tag{2.5}$$

For the simplicity of expressions, we introduce an auxiliary matrix

$$B = (b_{ij})_{N\times N} = \frac{1}{2}(A_t + A^2), \quad b_{ij} = \frac{1}{2}\left(a_{ij,t} + \sum_{k=1}^{N} a_{ik}a_{kj}\right),$$

and re-write the Eq. (2.5) into the form of components

$$Q_i(x_1, \ldots, x_N) \equiv -b_{it} - \sum_{k=1}^{N}(a_{ik}b_k + 2b_{ik}x_k) = \frac{\partial p}{\partial x_i}, \quad i = 1,2,\ldots,N. \tag{2.6}$$

In order to solve $p(\boldsymbol{x})$ from (2.6), these N equations should be compatible each other, that is, the vector functions $(Q_1, Q_2, \ldots, Q_N)$ should constitute a potential filed of scalar field p, whose sufficient and necessary conditions are

$$\frac{\partial Q_j(x_1, \ldots, x_N)}{\partial x_i} = \frac{\partial Q_i(x_1, \ldots, x_N)}{\partial x_j}, \quad i,j = 1,2,\ldots,N, \tag{2.7}$$

which hold if and only if

$$b_{ji} = b_{ij}, \quad i,j = 1,2,\ldots,N.$$