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Support vector machine adapted Tikhonov regularization method to solve Dirichlet problem

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ABSTRACT

Numerical solutions of partial differential equations are traditional topics that have been studied by many researchers. During the last decade, support vector machine (SVM) has been widely used for approximation problems. The contribution of this paper is two folds. One is to combine the reproducing kernel-SVM method with the Tikhonov regularization method, called the SVM-Tik methods, in which the kernels K_λ and K_λ^σ (see below) are newly developed. In the paper they are respectively phrased as the SVM-Tik-K_λ and SVM-Tik-K_λ^σ methods. The second contribution is to use the two models, SVM-Tik-K_λ and SVM-Tik-K_λ^σ, to solve the Dirichlet problem. The methods are meshless. They produce sparse representations in the linear combination form of specific functions (the K_λ and K_λ^σ kernels). The generalization bound result in learning theory is used to give an estimation of the approximation errors. With the illustrative examples the sparseness and robustness properties, as well as the effectiveness of the methods are presented. The proposed methods are compared with currently the most commonly used finite difference method (FDM) showing promising results.

1. Introduction

Many methods have been developed so far for solving partial differential equations. Some methods such as the finite difference method (FDM) [15] and the finite element method (FEM) [3] require the definition of a mesh (domain discretization) where the functions are approximated locally. The construction of a mesh in two or more dimensions is a non-trivial problem. A major disadvantage of those methods, however, is their mesh-dependent characteristics which normally require enormous computational effort and induce numerical instability when large number of grids or elements are required.

Another approach for solving partial differential equations is to use artificial neural networks (ANNs) [4–6]. The approach using ANNs to solve partial differential equations relies on the functional approximation capability of feedforward neural networks and results in construction of a solution written in a differentiable and closed analytic form. This form employs feedforward neural network as the basic approximation element, whose parameters (weights and biases) are adjusted to minimize an appropriate error function. The solution in terms of artificial neural networks posses several attractive features. One of the features is that the solution is infinitely differentiable and closed analytic form which can be easily used in any subsequent calculation. Another is that they posses smaller number of parameters compared to other solution technique

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[4,5]. However, ANNs suffer from their theoretical weakness. For example, back-propagation may not converge to a optimal global solution.

SVM, developed by Vapnik and his coworkers in 1995 [16], is based on statistical learning theory which seeks to minimize an upper bound of the generalization error consisting of the sum of the training error and a confidence interval. This principle is different from the commonly used empirical risk minimization (ERM) principle which only minimizes the training error. Based on this, SVMs usually achieve higher generalization performance than ANNs which implement ERM principle. As consequence, SVMs can be used wherever that ANNs can, and usually achieve better results. Another key characteristic of SVM is that training SVM is equivalent to solving a linearly constrained quadratic programming problem so that the solution of SVM is unique and global, unlike ANNs' training which requires nonlinear optimization with the possibility of getting stuck into local minima.

In this paper, we combine SVM with the Tikhonov regularization method which is called the SVM-Tik methods to solve Dirichlet problem numerically. Two kernels K_λ and K_λ^σ which are newly developed will be used in our algorithms, the corresponding algorithms are phrased as SVM-Tik-K_λ and SVM-Tik-K_λ^σ algorithms. The solutions are sparse representations in the linear combination form of specific functions (the K_λ and K_λ^σ kernels).

Experiments are done for testing the proposed approach. It shows good performance in noise-free data case and in Gaussian noise-corrupted data case. The comparisons are between three types of algorithms in Example 1. (1) The proposed SVM-Tik with K_λ kernel algorithm (SVM-Tik-K_λ). (2) The proposed SVM-Tik with K_λ^σ kernel algorithm (SVM-Tik-K_λ^σ). (3) The FDM. Although the FDM can achieve better performance than the two SVM based algorithms for noise-free data, the two SVM based algorithms behave better than the FDM in presence of Gaussian noise. With the two SVM based algorithms, sparse representations in linear combination form of specific functions (the K_λ and K_λ^σ kernel) are obtained. On the other hand, in the FDM case, the solution is not expressed in any closed analytical form as in our case, additional interpolation computations are required in order to find the value of the solution at particular points in the domain. As for comparison between the two SVM based algorithms, SVM-Tik-K_λ performs better than SVM-Tik-K_λ^σ.

2. Preliminary

We shall propose a new approach for constructing approximate solutions for the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = g, & \text{on } \partial D, \end{cases} \tag{2.1}$$

on an appropriate domain D in $\mathbf{R}^n$ with boundary ∂D, where $\mathbf{R}^n$ is the Euclidean space.

In our case, we shall first consider this problem in the Sobolev space $H^s(\mathbf{R}^n)$ with $n \geqslant 1,\ s \geqslant 2,\ s > n/2$ from the viewpoint of numerical analysis. In the sequel we abbreviate $H^s(\mathbf{R}^n)$ as H^s. The Sobolev Hilbert space H^s comprises functions F on $\mathbf{R}^n$ with the norm

$$\|F\|_{H^s}^2 = \int_{\mathbf{R}^n} |\hat{F}(\xi)|^2 (1+|\xi|^2)^s d\xi,$$

which admits a reproducing kernel

$$K_s(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{(1+|\xi|^2)^s} e^{i(x-y)\cdot\xi} d\xi, \tag{2.2}$$

where $\hat{F}$ is the Fourier transform of F,

$$\hat{F}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-i\xi\cdot x} F(x) dx$$

(See [7]).

The Dirichlet principle asserts that the solution of the Eq. (2.1) is the extremal function minimizing the Dirichlet integral under the boundary condition $F(x) = g(x)$ on ∂D.

We want to obtain some good representations of the extremal functions when they exist.

In [7] by applying the theory of Tikhonov regularization and the reproducing kernel based methods of Saitoh et al. in [2,8,10–13] the authors first formulated the problem as follows:

For a fixed $\lambda > 0$ and a given $g \in L_2(\partial D)$ find a solution for

$$\inf_{F\in H^s} \left\{ \lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \|F-g\|_{L_2(\partial D)}^2 \right\}. \tag{2.3}$$

The authors are considering the approximation by the Sobolev functions over the whole space.

The strategy is to first represent for each $\lambda > 0$ the extremal function $F = F^*_{s,\lambda,g}(x)$ in (2.3); and, secondly, to obtain the solution u of the problem (2.1) by taking the limit $\lambda \to 0$ on the corresponding extremal functions [7]. The theory of Tikhonov regularization guarantees that such limit exists.