Applied Mathematics and Computation 245 (2014) 520–525

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Existence of local solutions for a class of delayed neural networks with discontinuous activations

Mingjiu Gai *, Xiaolei Liu, Jian Yuan

Institute of System Science and Mathematics, Naval Aeronautical and Astronautical University, Yantai 264001, PR China

ARTICLE INFO

Keywords:
Local existence
Differential inclusion
Neural network
Filippov's solution

ABSTRACT

In this paper, we study the existence of local solutions for a class of differential inclusions, which right-hand set-valued functions may not be upper semicontinuous. Moreover, we employ our results to obtain the existence of local solutions for a class of neural networks.

1. Introduction

In the past decades, nonlinear dynamical systems described by differential equations or functional differential equations with discontinuous right-hand sides have been extensively studied, and successfully applied to various science and engineering fields such as mechanics, electrical engineering, automatic control, etc. As a special class of dynamical systems, discontinuous neural networks have received a great deal of attention in the literature recently. As far as we know, the paper [1] is the first one to deal with the global stability of a neural network with a discontinuous neuron activations. In the subsequent literature, considerable efforts have been devoted to investigate the neural network systems with discontinuous activation functions 2–16, etc.

It is well known that the existence of the solution is one of the basic questions for the dynamical neuron systems. Since the assumption of continuity on the activation functions is dropped, we need to specify a solution of the equation with a discontinuous right-hand side. The definition of such a solution which has been accepted universally is Filippov solution since a Filippov solution is a limit on the solutions of a sequence of ordinary differential equations with continuous right-hand sides. Therefore, the existence analysis of the solution in the sense of Filippov is an important step for understanding and designing discontinuous dynamical neuron systems, and enables us to study more complex dynamical behaviors such as periodic oscillation, stability, chaos, bifurcation, and so on.

In the literature [2], Forti, Nistri & Papini considered a class of neural networks described by the system of differential equations

$$\dot{x}(t) = -Dx(t) + Ag(x(t)) + A^{\tau}g(x(t-\tau)) + I, \tag{1}$$

where $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$ is the vector of neuron states, in which the symbol "$'$" denotes the transpose of a vector or a matrix; $D = \mathrm{diag}(d_1, \ldots, d_n)$ is an $n \times n$ constant diagonal matrix where $d_i > 0,\ i = 1, \ldots, n$, are the neuron self-inhibitions; $A = (a_{ij})$ and $A^{\tau} = (a^{\tau}_{ij})$ are $n \times n$ constant matrices which represent the neuron interconnection matrix and the delayed neuron interconnection matrix, respectively; and $\tau > 0$ is the constant delay in the neuron response; $I = (I_1, \ldots, I_n)' \in \mathbb{R}^n$ is the

* Corresponding author.
E-mail address: gaimingjiu@sina.com (M. Gai).

http://dx.doi.org/10.1016/j.amc.2014.06.025
0096-3003/

vector of constant neuron inputs. Moreover, $g(x) = (g_1(x_1), \dots, g_n(x_n))' : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal mapping where $g_i,\ i = 1, \dots, n$, represents the neuron input–output activation and satisfy the following assumption.

(H_0) Let $g_i \in \mathcal{G}$, for any $i = 1, \dots, n$, where $\mathcal{G}$ denotes the class of functions from $\mathbb{R}$ to $\mathbb{R}$ which are monotone nondecreasing and have at most a finite number of jump discontinuities in every compact interval.

Since the assumption of continuity on the activation functions is dropped, we need to specify a solution of the equation with a discontinuous right-hand side. Let

$$K[g(x)] = (K[g_1(x_1)], \dots, K[g_n(x_n)])',$$

where $K[g_i(x_i)] = \overline{\text{co}}[g_i(x_i)] = \left[g_i(x_i^-), g_i(x_i^+)\right]$. We define a solution of System (1) as follows.

Definition 1. For a continuous function $\phi(\theta) = (\phi_1(\theta), \dots, \phi_n(\theta))'$ and a measurable function $\psi(\theta) = (\psi_1(\theta), \dots, \psi_n(\theta))' \in K[g(\phi(\theta))]$ for a.a. $\theta \in [-\tau, 0]$, a continuous functions $x(t) = x(t; \phi, \psi) = (x_1(t), \dots, x_n(t))'$ associated with a measurable function $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))'$ is said to be a solution of the Cauchy problem for system (1) on $[-\tau, T)$ ($T > 0$ might be $+\infty$) with initial value $(\phi(\theta), \psi(\theta)),\ \theta \in [-\tau, 0]$, if $x(t)$ is absolutely continuous on any compact interval of $[0, T)$, and

$$\begin{cases} \dot{x}(t) = -Dx(t) + A\gamma(t) + A^\tau\gamma(t-\tau) + I, & \text{a.a. } t \in [0, T), \\ \gamma(t) \in K[g(x(t))], & \text{a.a. } t \in [0, T), \\ x(\theta) = \phi(\theta), & \theta \in [-\tau, 0], \\ \gamma(\theta) = \psi(\theta), & \theta \in [-\tau, 0] \end{cases} \tag{2}$$

holds for all $i = 1, 2, \dots, n$.

Any function γ in (2) is called an output solution associated to the state x. Observe that the above definition for the solution of System (1) implies that

$$\dot{x}(t) \in -Dx(t) + AK[g(x(t))] + A^\tau K[g(x(t-\tau))] + I, \quad \text{for a.a. } t \in [0, T). \tag{3}$$

Namely, $x(t)$ is a solution of System (1) in the sense of Filippov [17].

It is obviously that one solution of Inclusion (3) is not always a solution of System (1) by Definition 1. To solve this problem, Forti, Nistri & Papini put forward an interesting approach to obtain local existence of solutions for delayed neural networks with discontinuous activations [2], which is called "Step-by-Step Construction of a Local Solution", and have been employed in many subsequent literatures. The main idea of this method is described in the follows.

Fix a continuous initial function $\phi : [-\tau, 0] \to \mathbb{R}^n$, select a measurable function $\psi : [-\tau, 0] \to \mathbb{R}^n$ such that $\psi(s) \in K[g(\phi(s))]$ for a.a. $s \in [-\tau, 0]$, and consider the differential inclusion

$$\begin{cases} \dot{x}(t) \in -D(t)x(t) + AK[g(x(t))] + A^\tau\psi(t-\tau) + I, & \text{for a.a. } t \in [0, \tau], \\ x(0) = \phi(0). \end{cases}$$

By Filippov [17, p.77, Th.1], the inclusion has at least one solution x defined in a right neighborhood J of zero.

However, as noted in [11, p.1161, Remark 2.1], "the local existence x could not be obtained by the theorem [17, p.77, Th.1], since $\psi(t)$ is only a measurable function and the set-valued map

$$H(t, x) = -Dx + AK[g(x)] + A^\tau\psi(t-\tau) + I, \tag{4}$$

may not be upper semi-continuous due to the discontinuity of $\psi(t-\tau)$. So the result on the local existence of solutions in [2] needs to be improved."

We notice that the reason caused the above question is that the function (4) cannot satisfy Filippov's basic conditions which are requisite to Filippov's local existence theorem [17, p.77, Th.1]. To solve this problem, we attempt to extend Filippov's theorem by weakening its conditions. Our main approach is activated by the Filippov's approach also.

The rest paper is organized as follows: In Section 2, we study the local existence of solutions for a class of differential inclusions. In Section 3, we employ the new theorem in Section 2 to study the local existence of solutions for a class of recurrent neural networks. The conclusion is given in Section 4.

2. Local existence of solutions to a class of differential inclusion

In this section, we will investigate the local existence of solutions to a class of deferential inclusion

$$\dot{x} \in F(t, x) + f(t, x), \tag{5}$$

where $F(t, x)$ is a set-valued function, $f(t, x)$ is a single-valued function, and they satisfy the following hypotheses:

(H_1) $F(t, x)$ satisfies Filippov's basic conditions in the domain G, that is, for all $(t, x) \in G$, the set $F(t, x)$ is nonempty, bounded and closed, convex, and the function F is upper semicontinuous in t, x. (H_2) $f(t, x)$ is continuous except a countable set of jump discontinuous points in t, and continuous in x. Moreover, f is bounded measurable function in the domain G, i.e., there exists $m_0 > 0$, such that for all $(t, x) \in G$, $|f(t, x)| \leqslant m_0$. At a discontinuous point $(t, x) \in G$, we let $f(t, x) = f(t^-, x)$, where $f(t^-, x)$ is the left limit in t.