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Two generalizations of Lucas sequence

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ABSTRACT

We define a generalization of Lucas sequence by the recurrence relation $l_m = bl_{m-1} + l_{m-2}$ (if m is even) or $l_m = al_{m-1} + l_{m-2}$ (if m is odd) with initial conditions $l_0 = 2$ and $l_1 = a$. We obtain some properties of the sequence $\{l_m\}_{m=0}^{\infty}$ and give some relations between this sequence and the generalized Fibonacci sequence $\{q_m\}_{m=0}^{\infty}$ which is defined in Edson and Yayenie (2009). Also, we give corresponding generalized Lucas sequence with the generalized Fibonacci sequence given in Yayenie (2011).

1. Introduction

The Fibonacci sequence $\{F_m\}_{m=0}^{\infty}$ is the most famous sequence among integer sequences which is defined recursively by the relation $F_m = F_{m-1} + F_{m-2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers appear in many interesting places (see [2] for details). Another well-known sequence is the Lucas sequence $\{L_m\}_{m=0}^{\infty}$ which satisfies the same recurrence relation with the initial conditions $L_0 = 2$ and $L_1 = 1$.

The generating function for Fibonacci and Lucas sequences are, respectively,

$$\sum_{n=0}^{\infty} F_m x^m = \frac{x}{1-x-x^2} \quad \text{and} \quad \sum_{n=0}^{\infty} L_m x^m = \frac{2-x}{1-x-x^2},$$

and the Binet's formula for Fibonacci and Lucas sequences are, respectively,

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad \text{and} \quad L_m = \alpha^m + \beta^m,$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ are roots of the characteristic equation $x^2 - x - 1 = 0$. The positive root α is known as "golden ratio".

There are many generalizations of the Fibonacci sequence. One of them was given by Edson and Yayenie in [1] as follows;

$$q_0 = 0, \quad q_1 = 1, \quad q_m = \begin{cases} aq_{m-1} + q_{m-2}, & \text{if m is even} \\ bq_{m-1} + q_{m-2}, & \text{if m is odd} \end{cases} \quad (m \geqslant 2), \tag{1}$$

where a and b are two nonzero real numbers. They gave the generating function and some identities for the sequence $\{q_m\}_{m=0}^{\infty}$. Also they obtained the following extended Binet formula

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$$q_m = \left(\frac{a^{\xi(m+1)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right)\frac{\alpha^m - \beta^m}{\alpha - \beta}, \tag{2}$$

where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ are the roots of the characteristic equation $x^2 - abx - ab = 0$ and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function. Later, in [6], other relationships for the sequence $\{q_m\}_{m=0}^{\infty}$ were given. Some conditional sequences related to the sequence $\{q_m\}_{m=0}^{\infty}$ and their properties can be found in [1,3,4,6].

2. Generalized Lucas sequence

In this section, we define a generalization of the Lucas sequence similar to the generalized Fibonacci sequence $\{q_m\}_{m=0}^{\infty}$. We also give generating function and Binet's formula for this generalized sequence.

Definition 1. For any real nonzero numbers a and b, the generalized Lucas sequences $\{l_m\}_{m=0}^{\infty}$ is defined recursively by

$$l_0 = 2, \quad l_1 = a, \quad l_m = \begin{cases} bl_{m-1} + l_{m-2}, & \text{if m is even} \\ al_{m-1} + l_{m-2}, & \text{if m is odd} \end{cases} \quad (m \geqslant 2).$$

If we take $a = b = 1$, we have the classical Lucas sequence.

With the help of Lemma 3.1 in [5], we have the following identities.

Lemma 1. *The sequence $\{l_m\}_{m=0}^{\infty}$ satisfies the following properties*

$$l_{2n} = (ab+2)l_{2n-2} - l_{2n-4},$$
$$l_{2n+1} = (ab+2)l_{2n-1} - l_{2n-3}.$$

Theorem 1. *The generating function of the sequence $\{l_m\}_{m=0}^{\infty}$ is*

$$L(x) = \frac{2 + ax - (ab+2)x^2 + ax^3}{1 - (ab+2)x^2 + x^4}.$$

Proof. We define

$$L_0(x) = \sum_{m=0}^{\infty} l_{2m}x^{2m} \quad \text{and} \quad L_1(x) = \sum_{m=0}^{\infty} l_{2m+1}x^{2m+1}.$$

Note that

$$L_0(x) = 2 + (ab+2)x^2 + \sum_{m=2}^{\infty} l_{2m}x^{2m},$$
$$(ab+2)x^2L_0(x) = 2(ab+2)x^2 + \sum_{m=2}^{\infty}(ab+2)l_{2m-2}x^{2m},$$

and

$$x^4L_0(x) = \sum_{m=2}^{\infty} l_{2m-4}x^{2m}.$$

From the first equation in Lemma 1, we have

$$\left[1 - (ab+2)x^2 + x^4\right]L_0(x) = 2 - (ab+2)x^2,$$

and we get

$$L_0(x) = \frac{2 - (ab+2)x^2}{1 - (ab+2)x^2 + x^4}.$$

Similarly, we find

$$L_1(x) = \frac{a + ax^3}{1 - (ab+2)x^2 + x^4}.$$

Since $L(x) = L_0(x) + L_1(x)$, we obtain the desired result. □

The following theorem gives Binet's formula for generalized Lucas sequence $\{l_m\}_{m=0}^{\infty}$.