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Ostrowski type inequalities for functions whose higher order derivatives have a single point of non-differentiability

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ABSTRACT

Ostrowski type inequalities for the class of functions whose $(n-1)$th order derivatives are continuous, of bounded variation and have a single point of non-differentiability are derived. Special attention is given to functions whose first derivative has a single point of non-differentiability. Improvements of some previously obtained results are provided.

1. Introduction

The Ostrowski inequality (cf. [10]) states: if $f : [a,b] \to \mathbb{R}$ is a differentiable function with a bounded derivative, then

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right| \leqslant \left[\frac{b-a}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{b-a}\right]\|f'\|_\infty, \quad x \in [a,b]. \tag{1}$$

This inequality has been studied extensively during the years and many generalizations and refinements have been obtained. For some recent results, see for example [3,6,9,12] and the references therein.

The main objective of this paper is to give an Ostrowski type inequality for the following class of functions.

Definition 1. Let $x_0 \in [a,b] \subset \mathbb{R}$. A function $f : [a,b] \to \mathbb{R}$ is said to belong to the class $\mathcal{D}(x_0)$, that is, $f \in \mathcal{D}(x_0)$, if f is continuous on $[a,b]$, differentiable on $(a,x_0) \cup (x_0,b)$ and such that

$$M_L = \sup_{x\in(a,x_0)} |f'(x)| < +\infty \quad \text{and} \quad M_R = \sup_{x\in(x_0,b)} |f'(x)| < +\infty.$$

In the case $x_0 = a$ (resp. $x_0 = b$), we set $M_L = 0$ (resp. $M_R = 0$).

This class of functions has been introduced in [11]. As a motivation, the following class of functions was mentioned: the class of continuously differentiable functions on $[a,b]$ which are monotonous and convex or concave on $[a,x_0]$ and on $[x_0,b]$. For example, one can consider an increasing convex–concave function f with an inflexion point x_0. In the same paper, the following theorem was proved.

Theorem 1. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable on* Int$I$*, and let $[a,b] \subset$* Int$I$*. Suppose that $f' \in \mathcal{D}(x_0)$ for some $x_0 \in [a,b]$. Then, for $x \in [a,b]$, we have the following inequality*

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$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) + \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \leqslant S(p), \tag{2}$$

where

$$S(p) = \begin{cases} \frac{b-a}{2(p+1)^{1/p}(q+1)^{1/q}}\left(M_L^p \frac{(x_0-a)^{p+1}}{b-a} + M_R^p \frac{(b-x_0)^{p+1}}{b-a}\right)^{1/p}, & \text{if } 1 < p < \infty, \\ \frac{1}{4}\left(M_L(x_0-a)^2 + M_R(b-x_0)^2\right), & \text{if } p = 1, \\ \frac{b-a}{4}\max\{M_L(x_0-a), M_R(b-x_0)\}, & \text{if } p = \infty \end{cases}$$

and (p,q) *is a pair of conjugate exponents, that is,* $1/p + 1/q = 1$.

Note that $\lim_{p\to 1} S(p) = S(1)$, $\lim_{p\to\infty} S(p) = S(\infty)$ and also $S(p) \leqslant S(\infty)$ for $1 \leqslant p < \infty$ (cf. [11]). The aim of this paper is foremost to give an improvement of Theorem 1 for the case $p = 1$. Furthermore, we will also consider a more general case – the case when it is assumed that $f^{(n-1)} \in \mathcal{D}(x_0)$ for some $n \geqslant 1$. Similar results can be found in [2,7]. The proofs of our results rely heavily on the extended Euler formula, derived in [4].

Theorem 2. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is continuous and of bounded variation on* $[a,b]$ *for some* $n \geqslant 1$. *Then, for every* $x \in [a,b]$, *we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)dt - f(x) + \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\ &= \frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t), \end{aligned} \tag{3}$$

where $B_k(t)$ *is the kth Bernoulli polynomial and* $B_k^*(t) = B_k(t - \lfloor t \rfloor)$, $t \in \mathbb{R}$.

Since Bernoulli polynomials play an important role here, let us recall some of their basic properties. They are uniquely determined by

$$B_k'(x) = kB_{k-1}(x), \quad B_k(x+1) - B_k(x) = kx^{k-1}, \quad k \geqslant 0, \ B_0(x) = 1.$$

For the kth Bernoulli polynomial we have $B_k(1-x) = (-1)^k B_k(x)$, $x \in [0,1]$, $k \geqslant 1$. The first three Bernoulli polynomials are $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$ and $B_3(x) = x^3 - 3x^2/2 + x/2$.

$B_k^*(x)$ are periodic functions of period 1 such that $B_k^*(t) = B_k(t - \lfloor t \rfloor)$, $t \in \mathbb{R}$. For $k \geqslant 2$, $B_k^*(x)$ are continuous, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer.

The kth Bernoulli number B_k is defined by $B_k = B_k(0)$. For $k \geqslant 2$, we have $B_k(1) = B_k(0) = B_k$. Note that $B_{2k-1} = 0$ for $k \geqslant 2$, while $B_1(0) = -B_1(1) = -1/2$. For further details on Bernoulli polynomials see [1,8].

In what follows, $\|f\|_p^{[a,b]}$ stands for the L^p norm

$$\|f\|_p^{[a,b]} = \begin{cases} \left(\int_a^b |f(t)|^p dt\right)^{1/p}, & 1 \leqslant p < \infty \\ \operatorname{ess\,sup}_{t\in[a,b]} |f(t)|, & p = \infty. \end{cases}$$

2. Main results

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is of bounded variation on* $[a,b]$ *and* $f^{(n-1)} \in \mathcal{D}(x_0)$ *for some* $x_0 \in [a,b]$ *and* $n \geqslant 1$. *Then, for* $x \in [a,b]$, *we have*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) + \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]\right| \\ &\leqslant \begin{cases} \frac{(b-a)^{n-1}}{n!}\left(M_L^p(x_0-a) + M_R^p(b-x_0)\right)^{1/p}\|K_n(x,\cdot)\|_q^{[a,b]}, & 1 \leqslant p < \infty, \\ \frac{(b-a)^{n-1}}{n!}\max\{M_L, M_R\}\|K_n(x,\cdot)\|_1^{[a,b]}, & p = \infty, \end{cases} \end{aligned} \tag{4}$$

where (p,q) *is a pair of conjugate exponents, that is,* $1/p + 1/q = 1$, *and*

$$K_n(x,t) = B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right). \tag{5}$$

Proof. Starting from the right-hand side of (3) and applying the triangle inequality, the integral Hölder inequality and the discrete Hölder inequality, respectively, for $1 \leqslant p, q < \infty$, gives: