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A note on permanence of nonautonomous cooperative scalar population models with delays

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ABSTRACT

For a large family of nonautonomous scalar-delayed differential equations used in population dynamics, some criteria for permanence are given, as well as explicit upper and lower bounds for the asymptotic behavior of solutions. The method described here is based on comparative results with auxiliary monotone systems. In particular, it applies to a nonautonomous scalar model proposed as an alternative to the usual delayed logistic equation.

1. Introduction

In [3], Bastinec et al. studied the permanence of the following scalar nonautonomous delay differential equation (DDE) with a quadratic nonlinearity:

$$\dot{x}(t) = \sum_{k=1}^{m} \alpha_k(t)x(t-\tau_k(t)) - \beta(t)x^2(t), \quad t \geqslant 0, \tag{1.1}$$

where m is a positive integer, $\alpha_k,\ \beta : [0,\infty) \to (0,\infty)$ are continuous, $\tau_k : [0,\infty) \to [0,\infty)$ are continuous and uniformly bounded, $0 \leqslant \tau_k(t) \leqslant \tau$ for some $\tau > 0$, for $k = 1,\dots,m$, $t \geqslant 0$.

In view of the biological interpretation of model (1.1), only positive (or nonnegative) solutions of (1.1) are meaningful. In [3], the authors restrict their attention to solutions of (1.1) with initial conditions of the form

$$x(\theta) = \varphi(\theta), \quad -\tau \leqslant \theta \leqslant 0, \tag{1.2}$$

for $\varphi : [0,\infty) \to (0,\infty)$ continuous, and added the contraint $\sum_{k=1}^{m}\tau_k(t) > 0$ for all $t \geqslant 0$. Using the positivity of the functions $\alpha_k(t)$, $\beta(t)$, it is easy to see that solutions of (1.1)–(1.2) are positive whenever they are defined.

In a previous paper [2], the same authors considered a simpler nonautonomous model

$$\dot{x}(t) = r(t)\left[\sum_{k=1}^{m} \alpha_k x(t-\tau_k(t)) - \beta x^2(t)\right], \quad t \geqslant 0, \tag{1.3}$$

where the delay functions $\tau_k(t)$ satisfy all the conditions above, $r(t)$ is continuous and satisfies $r(t) \geqslant r_0$, $t \geqslant 0$, for some constant $r_0 > 0$, and α_k, β are positive constants, $1 \leqslant k \leqslant m$.

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Model (1.1) is a generalization of the DDE (1.3), obtained by considering a more general form of nonautonomous coefficients. The scalar DDE (1.3) has a positive equilibrium $K^* = \frac{1}{\beta}\sum_{k=1}^m \alpha_k$, which was proven in [2] to be a global attractor of all its positive solutions without any further restriction. In general, (1.1) does not have a positive equilibrium, so criteria for either extinction – when zero is a global attractor – or persistence or permanence play a crucial role.

Here, we set some standard notations. For (1.1) and for the DDEs hereafter, $C := C([-\tau, 0]; \mathbb{R})$ $(\tau > 0)$ with the usual sup norm $\|\varphi\|_\infty = \sup_{\theta\in[-\tau,0]}|\varphi(\theta)|$ will be taken as the phase space. For an abstract DDE in C,

$$\dot{x}(t) = f(t, x_t), \quad t \geqslant t_0, \tag{1.4}$$

where $f : \Omega \subset \mathbb{R} \times C \to \mathbb{R}$ is continuous, x_t denotes segments of solutions in C, $x_t(\theta) = x(t+\theta)$, $-\tau \leqslant \theta \leqslant 0$. If the solutions of initial value problems are unique, $x(t; t_0, \varphi)$ designates the solution of $\dot{x}(t) = f(t, x_t)$, $x_{t_0} = \varphi$; we use simply $x(t; \varphi)$ for $x(t; 0, \varphi)$. Even if it is not stated, we shall always assume that f is smooth enough so that initial value problems associated with (1.4) have unique solutions, with continuous dependence on data. This is the case if $f(t, \varphi)$ is uniformly Lipschitz continuous on the variable $\varphi \in C$ on each compact subset of Ω. For $C^+ := \{\varphi \in C : \varphi(\theta) \geqslant 0 \text{ for } -\tau \leqslant \theta \leqslant 0\}$, initial conditions (1.2) are written in the simpler form $x_0 = \varphi$ with $\varphi \in int(C^+)$. Cf. e.g. [5], for the concept of permanence given below, as well as for other standard definitions.

Definition 1.1. The scalar DDE (1.4) is said to be **permanent** (in $S = int(C^+)$, or another $S \subset C^+ \setminus \{0\}$) if there are positive constants m_0, M_0 with $m_0 < M_0$ such that, given any $\varphi \in S$, there exists $t_* = t_*(\varphi)$ such that $m_0 \leqslant x(t, \varphi) \leqslant M_0$ for $t \geqslant t_*$.

A nice criterion for the permanence of (1.1) was established in [3], assuming only that the functions $\alpha_k(t)$, $\beta(t)$ are uniformly bounded from above and from below by positive constants.

Theorem 1.1. [3] *Assume that* $\sum_{k=1}^m \tau_k(t) > 0$ *for all* $t \geqslant 0$. *If*
(h1) there are positive constants $\alpha_0, A_0, \beta_0, B_0$ *such that*

$$\alpha_0 \leqslant \alpha_k(t) \leqslant A_0, \quad \beta_0 \leqslant \beta(t) \leqslant B_0 \text{ for } t \geqslant 0, \quad k = 1, \ldots, m,$$

then the solutions of the initial value problems (1.1)–(1.2) are positive and defined on $[0, \infty)$, *and (1.1) is permanent in* $int(C^+)$. *Moreover, for every solution* $x(t)$ *of (1.1)–(1.2) the estimates*

$$m_0 \leqslant \liminf_{t\to\infty} x(t) \leqslant \limsup_{t\to\infty} x(t) \leqslant M_0, \tag{1.5}$$

hold with

$$m_0 = \liminf_{t\to\infty} \frac{1}{\beta(t)}\sum_{k=1}^m \alpha_k(t), \quad M_0 = \limsup_{t\to\infty} \frac{1}{\beta(t)}\sum_{k=1}^m \alpha_k(t). \tag{1.6}$$

The proof of this result in [3] is broken into several steps, and takes little advantage of the criterion established previously by the authors in [2]. Here, we present an alternative proof based on the fact that Eqs. (1.1) and (1.3) satisfy the quasimonotone condition. In fact, we shall show later (cf. Theorem 3.2) that we need not assume that $\sum_{k=1}^m \tau_k(t) > 0$ for all $t \geqslant 0$, and that initial conditions may be taken in the larger set $C_0 := \{\varphi \in C^+ : \varphi(0) > 0\}$. We recall that a scalar DDE (1.4) satisfies the **quasimonotone condition** (on the cone C^+) if for any $t \geqslant t_0$ and $\varphi, \psi \in C^+$ with $\varphi \leqslant \psi$ and $\varphi(0) = \psi(0)$, then $f(t, \varphi) \leqslant f(t, \psi)$ (cf. [7], p. 78). Under this condition, the semiflow is monotone. If $d_\varphi f(t, \varphi)$ exists, is continuous on $[t_0, \infty) \times C^+$, and $d_\varphi f(t, \varphi)\psi \geqslant 0$ for $\varphi, \psi \in C^+$ and $\psi(0) = 0$, then (1.4) is **cooperative**; cooperative equations satisfy the quasimonotone condition. Here, we abuse the terminology, and refer to equations satisfying the quasimonotone condition as *cooperative*.

Proof (*Alternative proof of Theorem 1.1*). Let $x(t) = x(t; \varphi)$ be the solution for an initial value problem (1.1)–(1.2), defined on some maximal interval $[0, a)$ with $a \in (0, \infty]$. Then $x(t)$ satisfies the inequality $\dot{x}(t) \leqslant A_0\sum_{k=1}^m x(t - \tau_k(t)) - \beta_0 x^2(t)$, $t \geqslant 0$. Comparing with the cooperative equation

$$\dot{u}(t) = A_0\sum_{k=1}^m u(t - \tau_k(t)) - \beta_0 u^2(t), \quad t \geqslant 0, \tag{1.7}$$

by Theorem 5.1.1 of [7] we have that $x(t)$ is bounded and defined on $[0, \infty)$, with $x(t) \leqslant u(t)$, $t \geqslant 0$, where $u(t)$ is the solution of (1.7)–(1.2). Moreover, by [2] the equilibrium $u^* = \frac{mA_0}{\beta_0}$ is a global attractor for all positive solutions of (1.7). We therefore conclude that $\limsup_{t\to\infty} x(t) \leqslant u^*$. In a similar way, we have $\dot{x}(t) \geqslant \alpha_0\sum_{k=1}^m x(t - \tau_k(t)) - B_0 x^2(t)$, $t \geqslant 0$, and by comparison with the cooperative equation

$$\dot{v}(t) = \alpha_0\sum_{k=1}^m v(t - \tau_k(t)) - B_0 v^2(t), \quad t \geqslant 0,$$

we obtain the lower bound $\liminf_{t\to\infty} x(t) \geqslant v^* := \frac{m\alpha_0}{B_0}$. Hence, (1.1) is permanent.

We now prove the estimates in (1.5),(1.6). Denote $\underline{x} = \liminf_{t\to\infty} x(t)$, $\bar{x} = \limsup_{t\to\infty} x(t)$. By the fluctuation lemma, there is a sequence (t_n), with $t_n \to \infty$ and $x(t_n) \to \bar{x}$, $\dot{x}(t_n) \to 0$. Fix a small $\varepsilon > 0$, and take $T > 0$ so that $x(t) \leqslant \bar{x} + \varepsilon$ for $t \geqslant T + \tau$. For n large enough so that $t_n - \tau_k(t_n) \geqslant T$ and $x(t_n) \geqslant \bar{x} - \varepsilon$, using (1.1) we get