Applied Mathematics and Computation 240 (2014) 140–148

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

High accuracy numerical methods for the Gardner–Ostrovsky equation

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ARTICLE INFO

Keywords:
Ostrovsky equation
Extended Korteweg–de Vries equation
Gardner equation
Solitons
Strong-stability-preserving method

ABSTRACT

The Gardner–Ostrovsky equation, also known as the extended rotation-modified Korteweg–de Vries (KdV) equation, describes weakly nonlinear internal oceanic waves under the influence of Earth' rotation. High accuracy numerical methods are needed to follow with precision the long time evolution of the solutions of this equation, with the additional difficulty that the numerical methods have to conserve very accurately several invariants of the solutions, including the mass and the energy of the waves. Finite-difference methods traditionally used for the solution of the KdV equation fails to preserve accurately these invariants for large times. In this paper we show that this difficulty can be overcome by using a high accuracy finite-difference (HAFD) numerical method. We present a strong-stability-preserving finite-difference scheme and compare its performance, using some relevant examples, with those of two recently published numerical methods for solving this kind of equation: a simpler second order finite-difference method and a pseudospectral numerical scheme that enforces the conservation of energy. The numerical comparison shows that the three methods have similar accuracy for short times, but the simpler finite-difference scheme is less accurate in preserving the three invariants, affecting the numerical accuracy of the solution as time goes on. On the contrary, the HAFD method presented here preserves the invariants with even better accuracy than the pseudospectral scheme, but with a much lower computational cost. In addition, the numerical implementation of the HAFD method is as easy as that of the simpler finite-difference method, being both much simpler than the intricate energy conservation pseudospectral scheme. These advantages makes the HAFD method presented here very appropriate for solving numerically this type of equations, particularly for studying long time wave propagation.

1. Introduction

We compare in this paper several numerical methods for solving the so-called Gardner–Ostrovsky (GO) equation, which in non-dimensional variables can be written as (see, e.g., [8,1,13])

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http://dx.doi.org/10.1016/j.amc.2014.02.003

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\alpha_1 u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}\right]=\gamma u, \tag{1.1}$$

where $\alpha,\ \alpha_1,\ \beta$ and γ are real non-dimensional parameters, under periodic boundary conditions of length L,

$$u(t,x)=u(t,x+L), \tag{1.2}$$

and initial condition

$$u(0,x)=u_0(x). \tag{1.3}$$

Eq. (1.1) differs from the original Ostrovsky equation [14] in the additional cubic nonlinearity (α_1 term), which becomes as important as the quadratic (α) term in the propagation of weakly nonlinear internal waves in the ocean under some relatively common circumstances (e.g., [8,1]). In turn, the Ostrovsky equation is a generalization of the well known Korteweg–de Vries (KdV) equation to include the effects of Earth's rotation on nonlinear dispersive internal waves in the ocean [14,8]. When $\gamma=0$, (1.1) represents the so-called Gardner's equation (see, e.g., [8,11]), or extended KdV equation that arose as a fundamental mathematical model for the description of weakly nonlinear dispersive waves in the situations when the higher-order nonlinearity effects (cubic term) become important [10]. Although (1.1) and similar equations were first derived in the context of the KdV equation to include different terms modeling additional physical effects relevant in the propagation of weakly nonlinear internal waves, they also describe some other quite different phenomena such as magneto-sonic waves in rotating plasma [12], waves in relaxing media [17], etc. (see, e.g., [7,6]).

We shall consider the general interval $-L_1\leqslant x\leqslant L_2$, with $L_1+L_2=L$, including the case in which L_1 and L_2 are very large (infinity) to allow for soliton-like solutions. The problem has three first integrals, similar to those of the Ostrovsky equation [3]:

$$I_1=\int_{-L_1}^{L_2} u\,dx=\text{constant}=0, \tag{1.4}$$

$$I_2=\int_{-L_1}^{L_2} u^2dx=\text{constant}, \tag{1.5}$$

$$I_3=\int_{-L_1}^{L_2} H\,dx=\text{constant}, \tag{1.6}$$

where H is the energy function or Hamiltonian, which is now given by

$$H=\frac{\alpha}{6}u^3+\frac{\alpha_1}{12}u^4-\frac{\beta}{2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\gamma}{2}\left(\partial_x^{-1}u\right)^2, \tag{1.7}$$

with ∂_x^{-1} the integral operator over x (see below). These first integrals of (1.1) and (1.2) correspond to the mass conservation (zero mean of u), the conservation of the L^2 norm of u, and the energy conservation, respectively.

Recently some conservative numerical schemes that takes into account these integrals properties (1.5) and (1.6) have been proposed by Yaguchi et al. [18] for the Ostrovsky equation. These authors considered both standard finite-difference and Fourier-pseudospectral schemes, showing that the energy conservative schemes perform better that the norm conservative schemes, specially when using a pseudospectral method. On the other hand, a much simpler finite-difference method has been recently proposed to solve the GO equation [13], but, as we shall show below, it does not preserve the above integral invariants as accurately as the energy conservative schemes, so that the precision of the solution worsens as time advances for the same spatial accuracy. The main aim of this paper is to show that a high accuracy finite-difference method, particularly, a sixth-order scheme for the discretization of the spatial derivatives combined with a fourth-order Runge–Kutta method for the time integration, is as accurate as the pseudospectral energy conservative scheme to preserve the integral invariants, but with a much smaller computational cost, and being also much simpler to implement numerically. The details of this numerical method are given in the next section. Section 3 compares the results obtained with the three numerical methods in two quite different examples. Special attention is paid to the conservation in time of the integral quantities (1.4)–(1.6) obtained by the different numerical methods. Finally, some conclusions are drawn in Section 4.

2. Numerical method

To integrate numerically Eq. (1.1), we first write it in the form

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\alpha_1 u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=\gamma\partial_x^{-1}u, \tag{2.8}$$

where

$$\partial_x^{-1}u\equiv\int_{-L_1}^{x}u(x',t)dx'-\frac{1}{L}\int_{-L_1}^{L_2}\left[\int_{-L_1}^{x}u(x',t)dx'\right]dx, \tag{2.9}$$