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Symmetrization, convexity and applications

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ABSTRACT

Based on permutation enumeration of the symmetric group and 'generalized' barycentric coordinates on arbitrary convex polytope, we develop a technique to obtain symmetrization procedures for functions that provide a unified framework to derive new Hermite–Hadamard type inequalities. We also present applications of our results to the Wright-convex functions with special emphasis on their key role in convexity. In one dimension, we obtain (up to a positive multiplicative constant) a method of symmetrization recently introduced by Dragomir (2014) [3], and also by El Farissi et al. (2012/2013) [4]. So our approach can be seen as a multivariate generalization of their method.

1. Introduction and preliminaries

Throughout Ω will always denote a convex polytope with a non-empty interior (that is, the convex hull of $(n+1)$ vertices $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ in $\mathbb{R}^d$). The points $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ may be regarded as vectors in any linear vector space, which in this paper will be taken to be Euclidean. We will refer to the vertex centroid of Ω or of $V(\Omega) := \{\boldsymbol{v}_i\}_{i=0}^n$ as the average of the vertices in $V(\Omega)$. We define the notion of (generalized) barycentric coordinates in the remainder of this paper as follows: let $\boldsymbol{x}$ be an arbitrary point of Ω. We call barycentric coordinates of $\boldsymbol{x}$ with respect to $V(\Omega)$ any set of real coefficients $\{\lambda_i(\boldsymbol{x})\}_{i=0}^n$ depending on the vertices of Ω and on $\boldsymbol{x}$ such that all the three following properties hold true:

$$\lambda_i(\boldsymbol{x}) \geqslant 0, \quad i = 0, \ldots, n, \tag{1}$$

$$\sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1, \tag{2}$$

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) \boldsymbol{v}_i. \tag{3}$$

The generalized barycentric coordinates for an arbitrary convex polytope are a key notion in formulating our method for symmetrization of functions. Recall that these coordinates exist for more general types of polytopes. The first result on their existence was due to Kalman [8, Theorem 2]. Barycentric coordinates for simplices are uniquely determined, however they can lose their uniqueness for general convex polytopes. One possible natural approach to constructing an interesting class of particular barycentric coordinates would be to simply construct a triangulation of the polytope Ω – the convex hull of the data set $V(\Omega)$ – into simplices such that the vertices $\boldsymbol{v}_i$ of the triangulation coincide with $V(\Omega)$. The fact that every convex

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polytope Ω can be triangulated using only the vertices of Ω is proven in the appendix of [2]. After that, one can use the standard barycentric coordinates for these simplices. (The above choice of coordinates is obviously immaterial, and it is only meant to simplify computations throughout and to fix the notation). For more details, see for instance [1]. As a result, each triangulation of $V(\Omega)$ generates a set of barycentric coordinates, that satisfy all the desirable properties (1)–(3). From now on, we assume that $\{\lambda_i\}_{i=0}^n$ is generated by a triangulation of $V(\Omega)$. In this situation, we list some other properties of these functions of which the following are particularly relevant to us:

(1) They are well-defined, piecewise linear and nonnegative real-valued continuous functions.
(2) Since vertices of a convex polytope are extremal points, it is easily deduced from (3) that $\{\lambda_i\}_{i=0}^n$ satisfy the delta property

$$\lambda_i(\boldsymbol{v}_j) = \delta_{ij}, \quad (i,j \in \{0,\ldots,n\}), \tag{4}$$

where we use Kronecker's delta.

We refer to reference [6] for details.

This paper is organized as follows. In Section 2, we establish and analyze links between permutation enumeration of the symmetric group and barycentric coordinates. In Section 3, we give our precise definition of symmetrization for any function defined on an arbitrary convex polytope. We will show that this method of symmetrization has some desirable properties. Under this method and the convexity assumption on the original function, our main result is that the resulting symmetrized function satisfies some new Hermite–Hadamard type inequalities. In Section 4, we impose a convexity assumption on the symmetrized function instead of the original function and present a refined version of our main results in this setting. Section 5 develops, under some assumptions on the polytope, a weighted general version of our method of symmetrization. The symmetrization techniques that we propose in this section are applicable to wide variety of domains, including simplices and Cartesian hyperrectangles. Finally, in Section 6, we consider some applications to the class of Wright-convex functions with special emphasis on their key role in convexity. In one dimension, we obtain (up to a positive multiplicative constant) a method of symmetrization recently introduced by Dragomir [3] and also by El Farissi et al. [4]. Some discussion of this is in Section 3. So our approach can be seen as a multivariate generalization of their method. This paper is motivated in part by the results presented in [3,4].

2. Permutations and barycentric coordinates

In this section, we establish and analyze links between permutations and barycentric coordinates on arbitrary convex polytopes. We will use them repeatedly in our further discussions.

We first start by presenting some basic notations and definitions. The set of all permutations on $\{0, 1, \ldots, n\}$ is denoted by S_n. Recall that each permutation $\sigma \in S_n$ is a 1-to-1 map:

$$\sigma : \{0,\ldots,n\} \to \{0,\ldots,n\},$$

so that $\{\sigma(0),\ldots,\sigma(n)\}$ is a re-arrangement of $\{0,\ldots,n\}$. There are $(n+1)!$ permutations in S_n, moreover, every permutation σ generates a mapping $T_\sigma : \Omega \to \mathbb{R}^d$, defined for all $\boldsymbol{x}$ in Ω by setting

$$T_\sigma(\boldsymbol{x}) := T_\sigma\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\boldsymbol{v}_i\right) = \sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\boldsymbol{v}_{\sigma(i)}.$$

The reader may check that T_σ is well-defined, continuous, and satisfies the property $T_\sigma(\Omega) \subset \Omega$, for any σ of S_n. In fact, we may say more about T_σ. Let us start by the following observations, which show how Ω, $T_\sigma(\Omega)$, $V(\Omega)$ and $V(T_\sigma(\Omega))$ are related for any $\sigma \in S_n$. Here, for any polytope X, we have used the notation $V(X)$ to denote the set of vertices of X.

Proposition 2.1. *For any permutation $\sigma \in S_n$, the mapping T_σ satisfies:*

$$T_\sigma(\boldsymbol{v}_j) = \boldsymbol{v}_{\sigma(j)}, \quad j = 0,\ldots,n, \tag{5}$$

$$T_\sigma(\Omega) = \Omega. \tag{6}$$

In particular, T_σ sends vertices of Ω to vertices of $T_\sigma(\Omega)$ and the vertex centroid of Ω is also that of $V(T_\sigma(\Omega)) := \{T_\sigma(\boldsymbol{v}_i)\}_{i=0}^n$. That is

$$\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i = \frac{1}{n+1}\sum_{i=0}^{n}T_\sigma(\boldsymbol{v}_i). \tag{7}$$

Proof. The first identity is simple to prove. It follows naturally from the Kronecker delta property of the barycentric coordinates. Indeed, by definition for all $j = 0,\ldots,n$, we have $T_\sigma(\boldsymbol{v}_j) = \sum_{i=0}^n \lambda_i(\boldsymbol{v}_j)\boldsymbol{v}_{\sigma(i)}$. Thus, the required equality follows immediately from the fact that $\lambda_i(\boldsymbol{v}_j) = \delta_{ij}$.