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Completely monotonic functions and inequalities associated to some ratio of gamma function

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ABSTRACT

Motivated by the work of Chen and Qi (2005) [3] we study the products $\prod_{k=1}^{n}\frac{3k-2}{3k}$ and $\prod_{k=1}^{n}\frac{3k-1}{3k}$. We prove that some functions associated to the previous products are completely monotonic and we establish some sharp inequalities.

1. Introduction

Chen and Qi [3] presented the following inequalities for the Wallis ratio for every natural number n:

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}}\leqslant\frac{(2n-1)!!}{(2n)!!}<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}},$$

where the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible. This inequality is a consequence of the complete monotonicity on $(0,\infty)$ of the function

$$x\mapsto\ln\frac{x\Gamma(x)}{\sqrt{x+\frac{1}{4}}\Gamma\left(x+\frac{1}{2}\right)}.$$

Here Γ is the gamma function given by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\quad (x>0).$$

In this paper we consider the following products for every integer $n\geqslant 1$:

$$P_1=\frac{1\cdot 4\ldots(3n-2)}{3\cdot 6\ldots(3n)},\quad P_2=\frac{2\cdot 5\ldots(3n-1)}{3\cdot 6\ldots(3n)}.$$

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The following representations in terms of gamma function

$$P_1 = \frac{\Gamma(n+\frac{1}{3})}{\Gamma(n+1)\Gamma(\frac{1}{3})}, \quad P_2 = \frac{\Gamma(n+\frac{2}{3})}{\Gamma(n+1)\Gamma(\frac{2}{3})} \tag{1}$$

motivate us to define the functions

$$x \mapsto \ln \frac{\left(\frac{1}{2\pi}\sqrt{3}\Gamma\left(\frac{2}{3}\right)\right)^3}{x^2\left(\frac{\Gamma\left(x+\frac{1}{3}\right)}{\Gamma(x+1)\Gamma\left(\frac{1}{3}\right)}\right)^3}$$

and

$$x \mapsto -\ln \frac{x\left(\frac{\Gamma\left(x+\frac{2}{3}\right)}{\Gamma(x+1)\Gamma\left(\frac{2}{3}\right)}\right)^3}{\frac{1}{\Gamma^3\left(\frac{2}{3}\right)}}.$$

In fact, we prove that these functions are completely monotonic on $(0,\infty)$. As a result, some sharp inequalities on P_1 and P_2 are established.

These results are improved in the final section of this work.

2. Estimates via complete monotonicity theory

The digamma function $\psi : [0,\infty) \longrightarrow \mathbb{R}$ is defined by the formula

$$\psi(x) = \frac{d}{dx}(\ln\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)},$$

while its derivatives $\psi', \psi'', \ldots$, are named polygamma functions. The following integral representations are of great help in this sections for every real number $x > 0$ and integer $n \geqslant 1$:

$$\psi^{(n)}(x) = (-1)^{n-1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}dt \tag{2}$$

and

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}dt. \tag{3}$$

See for details [1, p. 258].

Recall that a function s is completely monotonic on $(0,\infty)$ if it is indefinite derivable and $(-1)^n s^{(n)}(x) \geqslant 0$, for every real $x > 0$ and integer $n \geqslant 0$. A consequence of Hausdorff–Bernstein–Widder theorem (see [11]) states that a function s is completely monotonic on $(0,\infty)$ if and only if

$$s(x) = \int_0^\infty e^{-xt}\varphi(t)dt,$$

where φ is a non-negative function on $(0,\infty)$ such that the integral converges for all $x > 0$; see [11, p. 161].

Now we are in a position to give the following.

Theorem 1. *Let*

$$f(x) = \ln \frac{\left(\frac{1}{2\pi}\sqrt{3}\Gamma\left(\frac{2}{3}\right)\right)^3}{x^2\left(\frac{\Gamma\left(x+\frac{1}{3}\right)}{\Gamma(x+1)\Gamma\left(\frac{1}{3}\right)}\right)^3}.$$

Then f is completely monotone on $(0,\infty)$.

Proof. We have

$$f(x) = 3\ln\left[\frac{1}{2\pi}\sqrt{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\right] - 2\ln x - 3\ln\Gamma\left(x+\frac{1}{3}\right) + 3\ln\Gamma(x+1).$$