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All meromorphic solutions for two forms of odd order algebraic differential equations and its applications ☆

Wenjun Yuan [a,b,*], Yonghong Wu [c], Qiuhui Chen [d,*], Yong Huang [e]

[a] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
[b] Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, China
[c] Department of Mathematics and Statistics, Curtin University of Technology, GPO Box U 1987, Perth, WA 6845, Australia
[d] Cisco School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510420, China
[e] School of Computer Science and Education Software, Guangzhou University, Guangzhou 510006, China

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ABSTRACT

In this article, we employ the Nevanlinna's value distribution theory to investigate the existence of meromorphic solutions of algebraic differential equations. We obtain the representations of all meromorphic solutions for two classes of odd order algebraic differential equations with the weak $\langle p,q\rangle$ and dominant conditions. Moreover, we give the complex method to find all traveling wave exact solutions of corresponding partial differential equations. As an example, we obtain all meromorphic solutions of some generalized Bretherton equations by using our complex method. Our results show that the complex method provides a powerful mathematical tool for solving great many nonlinear partial differential equations in mathematical physics, and using the traveling wave nobody can find other new exact solutions for many nonlinear partial differential equations by any method.

1. Introduction and main results

In this paper, a meromorphic function $w(z)$ means that $w(z)$ is holomorphic in the complex plane $\mathbb{C}$ except for poles. $\wp(z;g_2,g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3. It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory, such as

$$T(r,f),\ m(r,f),\ N(r,f)\ \overline{N}(r,f),\ldots.$$

For detail of Nevanlinna's value-distribution theory, please see [1–3]. We denote by $S(r,f)$ any function satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to\infty$, possibly outside of a set of finite measure.

Recently, some authors find the exact solutions of certain PDEs combining with the complex ODEs. In 2013, Yuan et al. [4,5] researched the existence of meromorphic solutions of some algebraic differential equations with constant coefficients by using the Nevanlinna's value distribution theory, and gave the representations of all meromorphic solutions. At the same time, the complex method to find all traveling wave exact solutions of corresponding partial differential equations was given.

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* Corresponding authors. Address: Cisco School of Informatics, Guangdong University of Foreign Studies, Guangzhou 510420, China (Q. Chen).
E-mail addresses: wjyuan1957@126.com (W. Yuan), y.wu@curtin.edu.au (Y. Wu), chenqiuhui@hotmail.com (Q. Chen), yongh@gzhu.edu.cn (Y. Huang).

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In order to state these results, we need some concepts and notations:

Set $m \in \mathbb{N} := \{1, 2, 3, \ldots\},\ r_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ r = (r_0, r_1, \ldots, r_m),\ j = 0, 1, \ldots, m.$

$$M_r[w](z) := [w(z)]^{r_0}[w'(z)]^{r_1}[w''(z)]^{r_2} \cdots [w^{(m)}(z)]^{r_m}.$$

$\mathrm{d}(r) := r_0 + r_1 + \cdots + r_m$ is called the degree of $M_r[w]$.

Definition 1.1. A differential polynomial with constant coefficients is defined by

$$P[w] := \sum_{r \in I} a_r M_r[w],$$

where a_r are constants, and I is a finite index set. The total degree $\deg P[w]$ of $P[w]$ is defined by $\deg P[w] := \max_{r \in I}\{\mathrm{d}(r)\}$.

Consider the differential equations

$$E(z, w) := P[w] + w'w^{(m)} \mp \frac{1}{2}\left[w^{\left(\frac{m+1}{2}\right)}\right]^2 - aw^n = 0, \tag{1.1}$$

where $a \neq 0$ is a constant, $n \in \mathbb{N}$.

Definition 1.2. Let $p,\ q \in \mathbb{N}$. Suppose that the meromorphic solution w of the Eq. (1.1) has at least one pole. We say that the Eq. (1.1) satisfies $\langle p, q\rangle$ condition if there exactly exist p distinct meromorphic solutions of the Eq. (1.1) with pole of multiplicity q at $z = 0$. We say that the Eq. (1.1) satisfies weak $\langle p, q\rangle$ condition if substituting Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k, \quad q > 0,\ c_{-q} \neq 0 \tag{1.2}$$

into the Eq. (1.1), we can determine p distinct principle

$$\sum_{k=-q}^{-1} c_k z^k$$

with pole of multiplicity q at $z = 0$.

Definition 1.3. We say that a meromorphic function f belongs to the class W if f is an elliptic function, or a rational function of $e^{\alpha z},\ \alpha \in \mathbb{C}$, or a rational function of z.

Theorem 1.4 ([4,5]). *Let $p,\ l,\ m,\ n \in \mathbb{N}$, $\deg P[w] < n$, and the Eq. (1.1) satisfy $\langle p, q\rangle$ condition. Then all meromorphic solutions w of the Eq. (1.1) must be one of the following four cases:*

(I) *w is a constant.*

(II) *w is a rational function with $l(\leqslant p)$ distinct poles of multiplicity q, and $w := R(z)$ has the form of*

$$R(z) = \sum_{i=1}^{l}\sum_{j=1}^{q} \frac{c_{ij}}{(z - z_i)^j} + c_0. \tag{1.3}$$

(III) *w is a rational function $R(\xi)$ of $\xi = e^{\alpha z}(\alpha \in \mathbb{C})$. $R(\xi)$ has $l(\leqslant p)$ distinct poles of multiplicity q, and has the form (1.3).*

(IV) *w is an elliptic function with double periods $2\omega_1,\ 2\omega_2$, which has $l(\leqslant p)$ distinct poles of multiplicity q per parallelogram of periods. And w has one of the following three forms.*

(1.1) *If w is even, and the pole of $\wp(z)$ is the pole of w, then w is a rational $Q(\xi)$ in $\xi := \wp(z)$ and has the form of*

$$Q(\xi) = \sum_{i=1}^{l}\sum_{j=1}^{q} \frac{c_{ij}}{(\xi - \xi_i)^j} + \sum_{i=0}^{q_0} c_i \xi^i, \tag{1.4}$$

where $q_0 \leqslant \frac{q}{2}$. If the pole of $\wp(z)$ is not the pole of w, then $q_0 = 0$.

(1.2) *If w is odd, then $\frac{w}{\wp'(z)}$ is a rational $Q(\xi)$ of $\xi := \wp(z)$.*

(1.3) *If w is non-odd and non-even, then*

$$w(z) = Q_1(\wp(z)) + \wp'(z)Q_2(\wp(z)),$$

where $Q_1(\xi)$ and $Q_2(\xi)$ are rational functions with the form (1.4).