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# New fixed point results for maps satisfying implicit relations on ordered metric spaces and application

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ABSTRACT

We give some fixed point results for one and two mappings satisfying an implicit relation involving control function on ordered complete metric spaces. We furnish suitable examples to demonstrate the validity of the hypotheses of our results. We also give an application of our results to nonlinear Volterra integro-differential equations.



## 1. Introduction

Throughout this paper, we designate the set of all real nonnegative numbers by $\mathbb{R}^+$ and the set of all natural numbers by $\mathbb{N}$.

Fixed point theory is an important and actual topic of nonlinear analysis. Moreover, it is well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920 which was published in 1922 is one of the most important theorems in classical functional analysis. During the last four decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both. Due to the importance, generalizations of Banach fixed point theorem have been investigated heavily by many authors. This celebrated Banach contraction theorem can be stated as follow.

**Theorem 1** [11]. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{F}$ be a mapping of $\mathcal{X}$ into itself satisfying:*

$$d(\mathcal{F}x, \mathcal{F}y) \leqslant kd(x, y), \quad \forall\, x, y \in \mathcal{X}, \tag{1.1}$$

*where $k$ is a constant in $(0, 1)$. Then, $\mathcal{F}$ has a unique fixed point $x^* \in \mathcal{X}$.*

Inequality (1.1) implies continuity of $\mathcal{F}$. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

There is in the literature a great number of generalizations of the Banach contraction principle (see [26] and the references cited therein). In particular, obtaining the existence and uniqueness of fixed points for self-maps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. In this direction, Khan et al. [24] addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

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**Definition 1** (altering distance function [24]). $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(a) $\varphi$ is continuous and non-decreasing,
(b) $\varphi(t) = 0 \Longleftrightarrow t = 0$.

This notion has been used by several authors to establish fixed point results in a number of subsequent works, some of which are noted in [14,16,29,38].

On the other hand, in the recent years, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. Most of the results are based on a hybrid of two fundamental principles: order iterative technique and various contractive conditions. Indeed, they deal with a monotone (either order-preserving or order-reversing) mapping $\mathcal{F}$ satisfying, with some restriction, a classical contractive condition, and are such that for some $x_0 \in \mathcal{X}$, either $x_0 \preceq \mathcal{F}x_0$ or $\mathcal{F}x_0 \preceq x_0$ holds. The first result in this direction was given by Ran and Reurings [36] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [30] used the contractive condition

$$d(\mathcal{F}x, \mathcal{F}y) \leqslant kd(x, y) \quad \text{for } y \preceq x. \tag{1.2}$$

where $k \in [0, 1)$ and extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Later, in [35] O'Regan and Petruşel gave some existence results for Fredholm and Volterra type integral equations. In some of the above works, the fixed point results are given for non-decreasing mappings. Thereafter in [3], the authors used a nonlinear contractive condition, that is,

$$d(\mathcal{F}x, \mathcal{F}y) \leqslant \psi(d(x, y)) \quad \text{for } y \preceq x, \tag{1.3}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\lim_{n\to\infty}\psi^n(t) = 0$ for $t > 0$, instead of (1.2). Also in [3], the authors proved a fixed point theorem using generalized nonlinear contractive condition, that is,

$$d(\mathcal{F}x, \mathcal{F}y) \leqslant \psi\left(\max\left\{d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), \frac{1}{2}[d(x, \mathcal{F}y) + d(y, \mathcal{F}x)]\right\}\right) \tag{1.4}$$

for $y \preceq x$, where $\psi$ is as above.

Subsequently, Ćirić [15] generalized the results of Agarwal et al. [3], by introducing the concept of $\mathcal{S}$-monotone mapping and proved some fixed and common fixed point theorems for a pair of mappings satisfying $\mathcal{S}$-non-decreasing generalized nonlinear contractions in partially ordered complete metric spaces.

Recently, Altun and Simsek [4] proved fixed point results using implicit relation for one map and two maps and generalized the results given in [3,30,35,36]. Also an application for existence theorem for common solution of two integral equations is given. Here we recall their main results.

**Theorem 2.** *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered complete metric space. Suppose $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a non-decreasing mapping such that for all comparable $x, y \in \mathcal{X}$,*

$$T(d(\mathcal{F}x, \mathcal{F}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{F}y), d(x, \mathcal{F}y), d(y, \mathcal{F}x)) \leqslant 0$$

*where $T : \mathbb{R}_+^6 \to \mathbb{R}$ is a function as given in [4]. Also*

$$\mathcal{F} \text{ is continuous}$$

*or*

$$\begin{cases} \text{If } \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to x \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq x \text{ for all } n \end{cases}$$

*holds. If there exists an $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{F}x_0$ then $\mathcal{F}$ has a fixed point.*

**Theorem 3.** *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered complete metric space. Suppose $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{X}$ are two weakly increasing mappings such that for all comparable $x, y \in \mathcal{X}$,*

$$T(d(\mathcal{F}x, \mathcal{G}y), d(x, y), d(x, \mathcal{F}x), d(y, \mathcal{G}y), d(x, \mathcal{G}y), d(y, \mathcal{F}x)) \leqslant 0$$

*where $T : \mathbb{R}_+^6 \to \mathbb{R}$ is a function as given in Section 2. Also*

$$\mathcal{F} \text{ is continuous or } \mathcal{G} \text{ is continuous}$$

*or*

$$\begin{cases} \text{If } \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to x \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq x \text{ for all } n \end{cases}$$

*holds. Then $\mathcal{F}$ and $\mathcal{G}$ have a common fixed point.*