Applied Mathematics and Computation 240 (2014) 326–338

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Approximations of differentiable convex functions on arbitrary convex polytopes

Allal Guessab

Laboratoire de Mathématiques et de leurs Applications, UMR CNRS 4152, Université de Pau et des Pays de l'Adour, 64000 Pau, France

ARTICLE INFO

Keywords:
Barycentric approximation
Barycentric coordinates
Convex functions
Function approximation
Delaunay triangulation
Upper approximation operator

ABSTRACT

Let $X_n := \{\boldsymbol{x}_i\}_{i=0}^n$ be a given set of $(n+1)$ pairwise distinct points in $\mathbb{R}^d$ (called nodes or sample points), let $P = \mathrm{conv}(X_n)$, let f be a *convex* function with *Lipschitz continuous* gradient on P and $\lambda := \{\lambda_i\}_{i=0}^n$ be a set of barycentric coordinates with respect to the point set X_n. We analyze the error estimate between f and its barycentric approximation:

$$B_n[f](\boldsymbol{x}) = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) f(\boldsymbol{x}_i), \quad (\boldsymbol{x} \in P)$$

and present the best possible pointwise error estimates of f. Additionally, we describe the optimal barycentric coordinates that provide the best operator B_n for approximating f by $B_n[f]$. We show that the set of (linear finite element) barycentric coordinates generated by the Delaunay triangulation gives access to efficient algorithms for computing optimal approximations. Finally, numerical examples are used to show the success of the method.

1. Introduction, motivation and theoretical justification

We begin by considering the one-dimensional case since its simplicity allows us to analyse all the necessary steps through very simple computation. In the univariate approximation, say on an interval $[a, b]$, a simple way of approximating a given real function $f : [a, b] \to \mathbb{R}$ is to choose a partition $P := \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$, such that $a = x_0 < x_1 < \ldots < x_n = b$, and then to fit to f using a spline S_n of degree 1 at these points in such a way that:

1. The domain of S_n is the interval $[a, b]$;
2. S_n is a linear polynomial on each subinterval $[x_i, x_{i+1}]$;
3. S_n is continuous on $[a, b]$ and S_n interpolates the data, that is, $S_n(x_i) = f(x_i), \; i = 0, \ldots, n$.

This is a convenient class of interpolants because every such interpolant can be written in a barycentric form

$$S_n(x) = \sum_{i=0}^{n} \lambda_i(x) f(x_i), \quad (x \in [a, b]), \tag{1}$$

E-mail address: allal.guessab@univ-pau.fr

http://dx.doi.org/10.1016/j.amc.2014.04.075
0096-3003/

where

$$\lambda_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{if } x_{i-1} \leqslant x \leqslant x_i; \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{if } x_i \leqslant x \leqslant x_{i+1}; \\ 0, & \text{for all other } x. \end{cases}$$

Here, by a little abuse of notation, we set $x_{-1} := a$ and $x_{n+1} := b$. One of the main features of the usual linear spline approximation, in its simplest form (1), is that $\{\lambda_i\}_{i=0}^n$ form a (unique) set of (continuous) barycentric coordinates. This means that they satisfy, for all $x \in [a, b]$, three important properties:

$$\lambda_i(x) \geqslant 0, \quad i = 0, \dots, n;$$

$$\sum_{i=0}^{n} \lambda_i(x) = 1;$$

$$\sum_{i=0}^{n} \lambda_i(x) x_i = x.$$

This simple approach can be generalized to general polytopes. Throughout, we will assume all of the polytopes we work with are convex. Indeed, consider a given finite set of pairwise distinct points $X_n = \{\boldsymbol{x}_i\}_{i=0}^n$ in $P \subset \mathbb{R}^d$, with $P = \operatorname{conv}(X_n)$ denoting the convex hull of the point set X_n. We are interested in approximating an unknown scalar-valued continuous convex function $f : P \to \mathbb{R}$ from given function values $f(\boldsymbol{x}_0), \dots, f(\boldsymbol{x}_n)$ sampled at X_n. In order to obtain a simple and stable global approximation of f on P, we may consider a weighted average of the function values at data points of the following form:

$$B_n[f](\boldsymbol{x}) = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) f(\boldsymbol{x}_i), \tag{2}$$

or, equivalently, a convex combination of the data values $f(\boldsymbol{x}_0), \dots, f(\boldsymbol{x}_n)$. This means that we require that the system of functions $\lambda := \{\lambda_i\}_{i=0}^n$ forms a partition of unity, that is, for all $x \in P$, we have

$$\lambda_i(\boldsymbol{x}) \geqslant 0, \quad i = 0, \dots, n, \tag{3}$$

$$\sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1. \tag{4}$$

In addition, we shall also require the set of functions λ to satisfy the first-order consistency condition:

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) \boldsymbol{x}_i, \quad (\forall \boldsymbol{x} \in P). \tag{5}$$

We will call any set of functions $\lambda_i : P \to \mathbb{R}, \ i = 0, \dots, n$, *barycentric coordinates* if they satisfy the three properties (3)–(5) for all $\boldsymbol{x} \in P$. In view of these properties, we shall refer to the approximation schemes B_n as barycentric approximation (schemes). Barycentric coordinates also exist for more general types of polytopes. The first result on their existence was due to Kalman [15, Theorem 2]. It should be mentioned that one of the main difficulties in obtaining *all* barycentric approximations of functions, in dimensions higher than one, lies in the fact that their construction still remains a very difficult task in the general case. However, it should be emphasized, that as in the univariate case, one possible natural approach for constructing an interesting class of particular barycentric coordinates would be to simply construct a triangulation of the polytope P – the convex hull of the data set X_n – into simplices such that the vertices $\boldsymbol{v}_i$ of the triangulation coincide with $\boldsymbol{x}_i$. After that, one can use the standard barycentric coordinates for these simplices. As a result, each triangulation of the data set X_n generates a set of barycentric coordinates. Hence, there exists at least one barycentric approximation of type (2) which is generated by a triangulation. Let us outline shortly how triangulations and barycentric approximations are connected. It is known that every convex polytope can be triangulated into simplices, and the triangulation of a polytope may not be unique. To better illustrate this phenomenon, let us consider the simple example of a two-dimensional square S. Then two different triangulations are possible for S. Now every convex combination of the two associated coordinates provides a set of barycentric coordinates. This allows us to generate new families of barycentric approximations which are *not* generated by a triangulation. We refer to reference [4] for details.

A difficulty in minimizing the error estimate using the barycentric approximations arises from the possible existence of many barycentric coordinates. This yields the problem of selecting the barycentric coordinates as to minimize the approximation error. It will be interesting to have a way of selecting favourable ones among all barycentric approximations associated with the data set X_n.

Convex functions appear naturally in many disciplines of science such as physics, biology, medicine and economics, and they constitute an important part of mathematics. A natural question is: can these functions be well approximated by simpler functions and how?

While there are several papers investigating various methods to approximate arbitrary functions, very little research has been done subject to the usual convexity. For instance, if some smoothness is allowed for the function f which is to be