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An algorithmic method for checking global asymptotic stability of nonlinear polynomial systems with parameters

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ABSTRACT

An algorithm is presented here, for checking the global asymptotic stability of polynomial dynamical systems with parametric coefficients. It is based on the possibility of writing the polynomials, as sums of products of first degree polynomials, with artificial parametrical coefficients. By giving to all the parameters certain values, we ensure the positiveness of some quantities, constructing thereby proper Lyapunov functions, which guarantee the stability of the equilibrium point.

1. Introduction

It is widely known that when engineers and economists have to analyze mechanical, electrical or economical dynamic phenomena, they are usually dealing with nonlinear dynamical systems. They give attention to the stability of those systems and to some special dynamic properties they possess, [7]. Another issue of practical importance is whether the systems maintain their dynamic behavior as certain parameters are varied. Especially, it is interesting if a given equilibrium point remains stable.

In this paper we are working with autonomous polynomial dynamical systems of the form

$$\dot{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}, \boldsymbol{\mu}) \tag{1}$$

where, $\mathbf{x}$ is the state vector, consisting from functions of t, $\mathbf{\Phi}$ is a vector function, consisting from polynomial functions of elements of $\mathbf{x}$ and $\boldsymbol{\mu}$ a set of parameters. Our aim is to present an algorithm which detects values of $\boldsymbol{\mu}$ for which the equillibrium points of (1) are Global Asymptotic Stable. Actually, the said algorithm, discovers sets of values of the parameters which satisfy certain relations. These relations guarantee the stability behavior of the system around the equilibrium point.

There are a lot of efforts in the literature, for the description of proper algorithms which help us to face this problem. Let me refer to [9,8], to mention but a few. The algorithm presented here works as follows. First it accepts a polynomial function L as a Lyapunov function candidate, for the system (1), and an equilibrium point $\mathbf{x}_0$. Then it calculates the derivative of L across the trajectories of (1), denoted by $\dot{L}$. After that, it checks the positiveness of the quantities L and $-\dot{L}$. If this is true, under the assumption that the parameters satisfy a certain set of relations denoted by J, then we have stability of the equilibrium point.

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To explore the positiveness of the quantities L and $-\dot{L}$, we decompose them as follows:

$$\begin{aligned}V = {} & c_1(W_{i,\sigma,\varphi},\boldsymbol{\mu})[W_{1,-1,1}+x_1]^{j_{1,1}}\cdot[W_{2,-1,1}+W_{2,1,1}x_1+x_2]^{j_{2,1}} \\ & \cdot[W_{3,-1,1}+W_{3,1,1}x_1+W_{3,2,1}x_2+x_3]^{j_{3,1}}\dots[W_{n,-1,1}+W_{n,1,1}x_1+W_{n,2,1}x_2+\cdots+x_n]^{j_{n,1}}+\cdots \\ & +c_2(W_{i,\sigma,\varphi},\boldsymbol{\mu})[W_{1,-1,2}+x_1]^{j_{1,2}}\cdot[W_{2,-1,2}+W_{2,1,2}x_1+x_2]^{j_{2,2}} \\ & \cdot[W_{3,-1,2}+W_{3,1,2}x_1+W_{3,2,2}x_2+x_3]^{j_{3,2}}\dots[W_{n,-1,2}+W_{n,1,2}x_1+W_{n,2,2}x_2+\cdots+x_n]^{j_{n,2}}+\cdots \\ & +c_k(W_{i,\sigma,\varphi},\boldsymbol{\mu})[W_{1,-1,k}+x_1]^{j_{1,k}}\cdot[W_{2,-1,k}+W_{2,1,k}x_1+x_2]^{j_{2,k}} \\ & \cdot[W_{3,-1,k}+W_{3,1,k}x_1+W_{3,2,k}x_2+x_3]^{j_{3,k}}\dots[W_{n,-1,k}+W_{n,1,k}x_1+W_{n,2,k}x_2+\cdots+x_n]^{j_{n,k}}+R_{\mathcal{W}},\end{aligned} \tag{2}$$

where the exponents $j_{a,b}$ are specific positive whole numbers. The quantities $W_{i,\sigma,\varphi}$ are undetermined parameters that can take real values, the coefficients $c_j(W_{i,\sigma,\phi},\boldsymbol{\mu})$ and the quantity $R_{\mathcal{W}}$, called the remainder, are polynomial expressions of the parameters $W_{i,\sigma,\phi}$. We call these expressions the Linear-Like Factorizations of V. We obtain this "factorization" of the polynomials by means of a recursive algorithm, introduced in [2,6], which resembles to the Euclidean Algorithm, and annihilates successively the maximum terms. Then, we seek for those values of the parameters $W_{i,\sigma,\phi}$, which eliminate the non-square terms and make the coefficients of the square terms and the remainder, positive. Obviously, if this can be achieved, the positiveness of L and $-\dot{L}$ is secured and thus asymptotic stability is obtained. Moreover, the polynomial nature of the system implies also the satisfaction of the unbounded radius condition [5], which guarantees global asymptotic stability.

The main advantages of the method are:

(1) Our approach is symbolic in nature and not numeric.
(2) The calculations can be easily carried out, since the coefficients in the expression (2) have a specific construction. Each of them contains a number of parameters which is larger or equal than the number of the parameters of the previous coefficient. This triangular structure, more known as a sparse system of algebraic equations, permits their easier handling, [10].
(3) The values of the parameters $\boldsymbol{\mu}$, which guarantee stability, can be found straightforward, since they are involved in the calculations together with the artificial parameters $W_{i,\sigma,\phi}$, and therefore can be considered as polynomial functions of them.
(4) We are dealing with global asymptotic stability, not local.

We have to make clear that our method does not provide necessary and sufficient conditions. In other words, if our approach fails this does not mean that there are not values of the parameters which ensure stability or that another method could not find them.

Throughout this paper $\mathbf{R}$ will denote the set of real numbers.

2. Preliminaries

In this section we present the basic tools on which the basic algorithm is relied. Let $(x_1,x_2,\dots,x_n)$ be a vector of n-indeterminates which abbreviate by $\mathbf{x}$. An expression of the form $p=\sum_{\lambda=1}^{\varphi}c_\lambda x_1^{a_{1,\lambda}}x_2^{a_{2,\lambda}}\dots x_n^{a_{n,\lambda}}$, where $c_\lambda\in\mathbf{R}$ and some of the exponents $a_{i,j}\in\mathbf{Z}^+$ are not equal to zero, is called a polynomial in $x_1,x_2,\dots,x_n$ with real coefficients or, for short, a real polynomial. An element $x_1^{a_{1,\lambda}}x_2^{a_{2,\lambda}}\dots x_n^{a_{n,\lambda}}$ is called a *monomial* and an element $c_\lambda x_1^{a_{1,\lambda}}x_2^{a_{2,\lambda}}\dots x_n^{a_{n,\lambda}}$ is called a *term*. The quantity c_λ is the coefficient of the term. The sum $a_{1,\lambda}+a_{2,\lambda}+\cdots+a_{n,\lambda}$ is called *degree* of the term. A term is called *even* if all of its exponents are even integers, otherwise it is called an *odd* term. The term which corresponds to the exponent $(0,0,\dots,0)$, is the constant term. If we use the multi-index notation $\mathbf{a}_\lambda\in(\mathbf{Z}^+)^n$ to denote the vector $\mathbf{a}_\lambda=(a_{1,\lambda},a_{2,\lambda},\dots,a_{n,\lambda})$, we write a monomial compactly as $\mathbf{x}^{\mathbf{a}_\lambda}$ and a polynomial as $p=\sum_{\lambda=1}^{\varphi}c_\lambda\mathbf{x}^{\mathbf{a}_\lambda}$. The set of all real polynomials in $x_1,x_2,\dots,x_n$ is denoted by $\mathbf{R}[x_1,x_2,\dots,x_n]$ or $\mathbf{R}[\mathbf{x}]$. Let $\phi_{n,\lambda}=x_1^{a_{1,\lambda}}x_2^{a_{2,\lambda}}\dots x_n^{a_{n,\lambda}}$ and $\phi_{m,\mu}=x_1^{a_{1,\mu}}x_2^{a_{2,\mu}}\dots x_m^{a_{m,\mu}}$ be two monomials. We define the *lexicographical* order among monomials [1], as follows: we say that $\phi_{n,\lambda}$ is ordered less than $\phi_{m,\mu}$, denoted by $\phi_{n,\lambda}\prec\phi_{m,\mu}$, if either $n<m$ or $n=m$ and in the vector difference $\phi_{n,\mu}-\phi_{m,\lambda}$ the left-most nonzero entry is positive. In other words, the monomials are ordered as follows: $x_1\prec\cdots\prec x_1^7\prec\cdots\prec x_1x_2\prec\cdots\prec x_1x_2^8\prec\cdots\prec x_1x_2x_3\prec\cdots$ Let p be a given polynomial, ordered lexicographically, the term that corresponds to the maximum monomial is called the *maximum term* denoted by *maxterm*(p), its degree is called the polynomial degree and it is denoted by $\deg(p,\mathbf{x})$.

The next definitions will play a crucial role in the subsequents.

Definition 2.1. Let $\pi_i(\mathbf{x}),\ i=1,\dots,m$ be a collection of polynomials in $\mathbf{R}[\mathbf{x}]$. The set

$$V=V(\pi_i)=\{\boldsymbol{\theta}\in\mathbf{R}^n:\pi_i(\boldsymbol{\theta})=0,\ \text{for all } i=1,\dots,m\},$$

is called the *variety* defined by $\pi_i(\mathbf{x}),\ i=1,\dots,m$.