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A generalization of parameterized block triangular preconditioners for generalized saddle point problems

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ABSTRACT

In this paper, we consider a generalization of parameterized block triangular preconditioners for the generalized saddle point problems. The eigenvalue bounds for the preconditioned matrices are derived. Theoretical analysis shows that it slightly improves the existing results in the literature. Numerical experiments are given to show the efficiency of the GMRES with the generalized block triangular preconditioners.

1. Introduction

Consider the generalized saddle point problem

$$\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}(m \leqslant n)$ is of full row rank, and $C \in \mathbb{R}^{m\times m}$ is symmetric positive semi-definite, and $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are two column vectors. It is known that the linear system (1.1) arises in computational science and engineering applications, including computational fluid dynamics, constrained optimization, parameter identification, mixed finite element approximation of elliptic PDEs, and others [5,10,14,16,21]. Due to the coefficient matrix $\mathcal{A}$ is often large and sparse, iteration methods are more attractive than direct methods for solving system (1.1), see the detailed survey by Benzi et al. [5]. In the passed decades, as a classical iteration algorithm, Krylov subspace method has been widely applied, together with various preconditioners such as block diagonal preconditioners [11,17,18,21], block triangular preconditioners [6–8,12,22,26,27,29,30], constraint preconditioners [3,13,25], HSS preconditioners [1,2,4], matrix splitting preconditioners [23] and so on.

Recently, Simoncini [22] considered the indefinite block triangular preconditioner $\mathcal{P} = \begin{bmatrix} \hat{A} & B^T \\ O & -\hat{C} \end{bmatrix}$, together with a Krylov subspace iterative solver for (1.1). Here $\hat{A} \in \mathbb{R}^{n\times n}$ and $\hat{C} \in \mathbb{R}^{m\times m}$ are symmetric positive definite. Cao proposed in [6] the positive block triangular preconditioner $\mathcal{G} = \begin{bmatrix} \hat{A} & B^T \\ O & \hat{C} \end{bmatrix}$. The detailed eigenvalue analyses for $\mathcal{AP}^{-1}$ and $\mathcal{AG}^{-1}$ were shown in [22,6], respectively, and numerical results illustrated that both the block triangular preconditioners $\mathcal{P}$ and $\mathcal{G}$ are more efficient than the following block diagonal preconditioners [11,18]

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$$D_{-}=\begin{bmatrix}\hat{A} & O\\ O & -\hat{C}\end{bmatrix}\quad\text{and}\quad D_{+}=\begin{bmatrix}\hat{A} & O\\ O & \hat{C}\end{bmatrix}.$$

Some improved estimates for $\mathcal{A}\mathcal{P}^{-1}$ and $\mathcal{A}\mathcal{G}^{-1}$ were presented in [8,15]. Motivated by the idea that the convergence of the iteration methods such as GMRES may be improved by introducing ω to the block triangular preconditioners, Jiang et al. [12] studied two parameterized block triangular preconditioners

$$P_{-}=\begin{bmatrix}\hat{A} & \omega B^T\\ O & -\hat{C}\end{bmatrix}\quad\text{and}\quad P_{+}=\begin{bmatrix}\hat{A} & \omega B^T\\ O & \hat{C}\end{bmatrix}.$$

It was proved that the preconditioned matrices $\mathcal{A}P_{-}^{-1}$ and $\mathcal{A}P_{+}^{-1}$ have similar properties as $\mathcal{A}\mathcal{P}^{-1}$ and $\mathcal{A}\mathcal{G}^{-1}$.

In this paper, we are interested in the generalized block triangular preconditioners

$$P=\begin{bmatrix}\hat{A} & \omega B^T\\ O & j\hat{C}\end{bmatrix}, \tag{1.2}$$

where $\omega \geqslant 0$ and j is a nonzero real parameter. When $j=-1$ and $j=1$, the block triangular matrices P reduce to P_{-} and P_{+}, respectively. Particularly, the matrix P reduces to the block diagonal preconditioner denoted by

$$D=\begin{bmatrix}\hat{A} & O\\ O & j\hat{C}\end{bmatrix}, \tag{1.3}$$

when $\omega=0$. For clarity, we use P_1 and P_2 to denote the preconditioner P when j is positive or negative, respectively. We will give the eigenvalue analysis of the preconditioned matrices $\mathcal{A}P_1^{-1}$ and $\mathcal{A}P_2^{-1}$. Similar to [6], we will illustrate that all the eigenvalues of the preconditioned matrix $\mathcal{A}P_1^{-1}$ are real, and present a valid estimate on the bounds for the eigenvalues of $\mathcal{A}P_1^{-1}$. Theoretical analysis given in this paper shows that the derived bounds are very tight, and superior to the existing results in [6,12,15]. Similarly, the upper and lower bounds on the real part of the eigenvalues of the preconditioned matrix $\mathcal{A}P_2^{-1}$ are derived, which also improve the theoretical results in [12,28]. In particular, we will provide the bounds on the real and imaginary parts of all the complex eigenvalues of $\mathcal{A}P_2^{-1}$, which generalize the results in [28,30]. Numerical experiments associated with Stokes problem are presented to confirm the theoretical analysis, and illustrate the efficiency of the GMRES [20] with the generalized block triangular preconditioners.

Notation. We denote by $\|x\|=\sqrt{x^Tx}$ the Euclidean norm of a vector x, and diag (Λ_1,Λ_2) the block diagonal matrix with diagonal matrices Λ_1 and Λ_2. For an eigenvalue μ, we use $\mathcal{R}(\mu)$ and $\mathcal{I}(\mu)$ to denote its real and imaginary parts, respectively.

2. Eigenvalue analysis of the preconditioned matrix $\mathcal{A}P^{-1}$

We consider the eigenvalue problem

$$\mathcal{A}P^{-1}\begin{bmatrix}u\\ v\end{bmatrix}=\mu\begin{bmatrix}u\\ v\end{bmatrix}.$$

It can be equivalently represented as the generalized eigenvalue problem

$$\mathcal{A}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}=\mu P\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix} \tag{2.1}$$

with $\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}=P^{-1}\begin{bmatrix}u\\ v\end{bmatrix}$.

Denote $\hat{D}=\begin{bmatrix}\hat{A}^{\frac{1}{2}} & O\\ O & \sqrt{|j|}\hat{C}^{\frac{1}{2}}\end{bmatrix}$. Let $\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}=\hat{D}^{-1}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}$, then (2.1) can be rewritten as

$$\hat{D}^{-1}\mathcal{A}\hat{D}^{-1}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}=\mu\hat{D}^{-1}P\hat{D}^{-1}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix} \tag{2.2}$$

with

$$\hat{D}^{-1}\mathcal{A}\hat{D}^{-1}=\begin{bmatrix}\hat{A}^{-\frac{1}{2}}A\hat{A}^{-\frac{1}{2}} & 1\sqrt{|j|}^{-\frac{1}{2}}B^TC^{-\frac{1}{2}}\\ \frac{1}{\sqrt{|j|}}\hat{C}^{-\frac{1}{2}}B\hat{A}^{-\frac{1}{2}} & -\frac{1}{|j|}\hat{C}^{-\frac{1}{2}}C\hat{C}^{-\frac{1}{2}}\end{bmatrix},$$

$$\hat{D}^{-1}P\hat{D}^{-1}=\begin{bmatrix}I & \frac{\omega}{\sqrt{|j|}}\hat{A}^{-\frac{1}{2}}B^T\hat{C}^{-\frac{1}{2}}\\ O & \frac{j}{|j|}I\end{bmatrix}.$$