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Entire solutions of a lattice vector disease model

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ABSTRACT

In this article, the existence and qualitative properties of entire solutions for a lattice vector disease model is considered. These entire solutions are constructed by the combinations of traveling wave solutions and the spatial independent solution of the model, which reveal some new transmission forms of the disease.

1. Introduction

Since the pioneer work of Kermack and McKendrick [1,2], the investigation of disease models is one of the dominant themes in both epidemics and mathematical biology. Mathematically, these models include various of differential systems such as ordinary differential systems and reaction–diffusion systems, etc. (see [3–13]). Recently, some articles considered the disease spread in the patchy environment, see [7,9,12,14]. If the number of the patches is infinite, Xu and Weng [15] derived a disease spread model in a 1-dimensional lattice environment, as the best we know, which is the first lattice disease model considered in the literature. The model in Xu and Weng [15] is as follows:

$$\frac{du_n(t)}{dt} = d[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - au_n(t) + b[1 - u_n(t)]\int_0^{+\infty}\sum_{j\in\mathbb{Z}} F_j(s)u_{n-j}(t-s)ds, \tag{1.1}$$

where $u_n(t)$ denotes the density of an infectious host at time t and location $n \in \mathbb{Z}$, $d > 0$ is the diffusion coefficient, $a > 0$ is the cure/recovery rate of the infected host, and $b > a$ is a constant related to the host-vector contact rate. For more details of the derivation of this model, we refer to [15].

Actually, Eq. (1.1) is a kind of lattice models with infinite time delay and nonlocal reaction (see [16] for more information of lattice differential equations). In [15], they assumed the following conditions on $F_j(s)$:

(C1) $F_j \in C(\mathbb{R}^+ \times \mathbb{Z}, \mathbb{R}^+)$ with $F_j(s) = F_{-j}(s)$ and $F_j(s) \geqslant 0$ for all $j \in \mathbb{Z}$, $s \geqslant 0$, and $\int_0^{+\infty}\sum_{j\in\mathbb{Z}}F_j(s)ds = 1$;
(C2) For any $c \geqslant 0$, there is some $\lambda^\sharp := \lambda^\sharp(c) > 0$ such that

$$\int_0^{+\infty}\sum_{j\in\mathbb{Z}}F_j(s)e^{-\lambda(j+cs)}ds < +\infty \quad \text{for all } \lambda \in [0, \lambda^\sharp)$$

and $\lim_{\lambda\uparrow\lambda^\sharp}\int_0^{+\infty}\sum_{j\in\mathbb{Z}}F_j(s)e^{-\lambda(j+cs)}ds = +\infty$, where $\lambda^\sharp$ may be $+\infty$.

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Under these conditions, it is easy to see that Eq. (1.1) admits two equilibria $u \equiv 0$ and $u \equiv K := 1 - a/b$. Xu and Weng [15] derived the existence and isotropic properties of the initial value problem, the spreading speed and the existence of traveling wave solutions of Eq. (1.1).

We introduce the following proposition from [15] for the use of convenience.

Proposition 1.1. *Assume (C1) and (C2). Define $c_* > 0$ as follows:*

$$c_* = \inf\{c > 0 : L_c(\lambda) \leqslant 1 \text{ for some } \lambda \in (0, \lambda^\sharp]\},$$

where

$$L_c(\lambda) := \frac{1}{2d + a + c\lambda}\left[d(e^{\lambda} + e^{-\lambda}) + b\int_0^{+\infty}\sum_{j\in\mathbb{Z}} F_j(s)e^{-\lambda(j+cs)}ds\right].$$

Then for any $c \geqslant c_*$, Eq. (1.1) admits a positive traveling wave solution $u_n(t) = \phi_c(\xi)$, $\xi = n + ct$, which satisfies $\phi_c(-\infty) = 0$, $\phi_c(+\infty) = K$. Furthermore, for $c > c_*$, $\phi_c'(\xi) > 0$ on $\mathbb{R}$ and

$$\lim_{\xi\to-\infty}\phi_c(\xi)e^{-\lambda_1(c)\xi} = 1, \quad \lim_{\xi\to-\infty}\phi_c'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c).$$

Here $\lambda_1(c) > 0$ is the smallest solution to the characteristic equation

$$c\lambda - d(e^{\lambda} + e^{-\lambda} - 2) + a - b\int_0^{+\infty}\sum_{j\in\mathbb{Z}} F_j(s)e^{-\lambda(j+cs)}ds = 0.$$

It is known that the traveling wave solution is a kind of special and important solutions, which describes the travels of the solution from one steady state to another steady state without changes in shape. For entire solutions, we mean the solutions defined in the whole space and for the time $t \in \mathbb{R}$. Of cause, the traveling wave solution is also a kind of entire solutions. In the latest decades, there have been many papers devoted to the constructions of the entire solutions by the combinations of traveling wave solutions and spatially independent solution [17–24], since the pioneer works of Hamel and Nadirashvili [25,26].

The following lemma illustrates the existence of the spatially independent solution of Eq. (1.1).

Lemma 1.1. *Assume (C1) and (C2). Consider the following spatially independent equation of Eq. (1.1):*

$$\frac{du(t)}{dt} = -au(t) + b[1 - u(t)]\int_0^{+\infty}\sum_{j\in\mathbb{Z}} F_j(s)u(t-s)ds. \tag{1.2}$$

Then Eq. (1.2) admits a heteroclinic solution $\Gamma(t)$, which is increasing on $\mathbb{R}$ and satisfies

$$\lim_{t\to-\infty} e^{-\lambda_* t}\Gamma(t) = K, \quad \Gamma(+\infty) = K, \quad \Gamma(t) \leqslant Ke^{\lambda_* t} \quad \text{and} \quad \Gamma'(t) > 0 \quad \text{for } t \in \mathbb{R}.$$

Here λ_* is the positive root of the equation

$$\Lambda(\lambda) := \lambda + a - b\int_0^{+\infty}\sum_{j\in\mathbb{Z}} F_j(s)e^{-\lambda s}ds = 0. \tag{1.3}$$

Define $A_c > 0$ for each $c \in (c_*, +\infty)$ by

$$A_c := \inf\{A > 0 : A \geqslant \phi_c(\xi)e^{-\lambda_1(c)\xi} \text{ for any } \xi \in \mathbb{R}\}. \tag{1.4}$$

It is easy to see $A_c \geqslant \lim_{\xi\to-\infty}\phi_c(\xi)e^{-\lambda_1(c)\xi} = 1$.

This paper is devoted to both the existence and qualitative properties for entire solutions of Eq. (1.1). This kind of entire solutions show some new transmission forms of the disease. The following is the main result on the existence of the entire solutions of Eq. (1.1). The qualitative properties of the entire solutions will be presented in Section 5.

Since the solutions of Eq. (1.1) with initial values may not be twice differentiable, we further assume the following two conditions on $F_j(s)$:

(C3) There exists a positive constant $\bar{M} > 0$ such that $\int_0^{+\infty}\sum_{j\in\mathbb{Z}}|F_j(h+s) - F_j(s)|ds \leqslant \bar{M}h$ for any $h \in \mathbb{R}_+$.
(C4) There exists a positive constant $\tilde{M} > 0$ such that $\int_0^h\sum_{j\in\mathbb{Z}}F_j(s)ds \leqslant \tilde{M}h$ for any $h \in \mathbb{R}_+$.

Conditions (C3) and (C4) are used for the proof of the following Theorem 1.1, and the idea is motivated from [22]. The following remark shows that there does exit such kernel functions which satisfy (C1)–(C4).

Remark 1.1. For any $\tau > 0$ fixed, the kernel function

$$F_k(s) = \frac{1}{\tau}e^{-\frac{s}{\tau}}, \quad F_j(s) \equiv 0 \quad \text{for } j \neq k, \tag{2}$$

admits (C1)–(C4).