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Repeated derivatives of composite functions and generalizations of the Leibniz rule

D. Babusci [a], G. Dattoli [b], K. Górska [c,d,e,*], K.A. Penson [e]

[a] INFN – Laboratori Nazionali di Frascati, via E. Fermi, 40, I 00044 Frascati, Roma, Italy
[b] ENEA – Centro Ricerche Frascati, via E. Fermi, 45, I 00044 Frascati, Roma, Italy
[c] Instituto de Física, Universidade de São Paulo, P.O. Box 66318, BR 05315-970 São Paulo, SP, Brazil
[d] H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, ul. Eljasza-Radzikowskiego 152, PL 31342 Kraków, Poland
[e] Sorbonne Universités, Laboratoire de Physique Théorique de la Matière Condensée (LPTMC), Université Pierre et Marie Curie, CNRS UMR 7600, Tour 13 – 5ième ét., Boîte Courrier 121, 4 place Jussieu, F 75252 Paris Cedex 05, France

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ABSTRACT

We use the properties of Hermite and Kampé de Fériet polynomials to get closed forms for the repeated derivatives of functions whose argument is a quadratic or higher-order polynomial. These results are extended to product of functions of the above argument, thus giving rise to expressions which can formally be interpreted as generalizations of the familiar Leibniz rule. Finally, examples of practical interest are discussed.

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1. Introduction

Formula (1.1.1.1) of Ref. [1], i.e. ($\hat{D}_\xi = \mathrm{d}/\mathrm{d}\xi$)

$$\hat{D}_x^n[f(x^2)] = n! \sum_{k=0}^{[n/2]} \frac{(2x)^{n-2k}}{k!\,(n-2k)!} \hat{D}_{x^2}^{n-k} f(x^2), \tag{1.1}$$

refers to repeated derivatives of functions whose argument is a quadratic polynomial. Albeit elementary, and a particular case of the Faà di Bruno formula [2], some aspects of Eq. (1.1) deserve to be studied as they may lead to novel results.

Eq. (1.1) implicitly assumes that $f(x)$ is at least a n-times differentiable function. Moreover, we make the hypothesis that it can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f_n = \mathrm{e}^{x\hat{\phi}} f_0, \quad \hat{\phi}^n f_0 = f_n, \tag{1.2}$$

where f_0 denotes a kind of "vacuum" state on which the repeated action of the umbral operator $\hat{\phi}$ generates a discrete sequence of functions. The word "vacuum" is borrowed from the Quantum Mechanics, where the entire spectrum and the eigenfunctions of the Hamiltonian can sometimes be generated by a repeated action of a certain "creation" operator acting on the ground state (= "vacuum"), see Ref. [3]. In what follows we will consider only functions that admit the expansion (1.2), i.e., that are analytical over the entire complex plane.

* Corresponding author at: H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, ul. Eljasza-Radzikowskiego 152, PL 31342 Kraków, Poland.

E-mail addresses: danilo.babusci@lnf.infn.it (D. Babusci), dattoli@frascati.enea.it (G. Dattoli), k.gorska80@gmail.com (K. Górska), penson@lptl.jussieu.fr (K.A. Penson).

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As an example, we note that the Tricomi–Bessel function of order zero [4] can be written as

$$C_0(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{(k!)^2} = \mathrm{e}^{-x\,\hat{\phi}} f_0 \tag{1.3}$$

with

$$\hat{\phi}^k f_0 = \frac{1}{\Gamma(k+1)}, \quad f_0 = \frac{1}{\Gamma(1)} = 1. \tag{1.4}$$

The operator $\hat{\phi}$ may also be raised to any real (not necessarily integer) power, so that the Tricomi–Bessel function of order α can be written as

$$C_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\,\Gamma(k+\alpha+1)} = \hat{\phi}^\alpha\, \mathrm{e}^{-x\,\hat{\phi}} f_0. \tag{1.5}$$

The Tricomi–Bessel functions $C_n(x)$ introduced above are close relatives of Bessel functions $J_n(x)$, as the following formula shows:

$$J_n(x) = \left(\frac{x}{2}\right)^n C_n\left[\left(\frac{x}{2}\right)^2\right]. \tag{1.6}$$

Their use here is purely formal. Note however that for non-integer indices $n = \nu$ the $C_\nu(x)$ are free from any singularity at the origin. The obvious advantage of the previous umbral re-shaping of the function $f(x)$ is the possibility of exploiting the wealth of the properties of the exponential function, which will be assumed to be still valid, since we will treat the operator $\hat{\phi}$ as an ordinary constant [4].

As a consequence of Eq. (1.2), we can write (we omit the "vacuum" f_0 for brevity)

$$\hat{D}^n_{g(x)} f[g(x)] = \hat{\phi}^n\, \mathrm{e}^{g(x)\,\hat{\phi}}. \tag{1.7}$$

By taking into account the identity [8] (compare also Eq. (1.1.3.3) on p. 5 of [1])

$$\hat{D}^n_x\, \mathrm{e}^{a x^2} = H^{(2)}_n(2\,a x, a)\, \mathrm{e}^{a x^2}, \tag{1.8}$$

where

$$H^{(2)}_n(x,y) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}\, y^k}{(n-2k)!\,k!} \tag{1.9}$$

are the two-variable Hermite–Kampé de Fériet polynomials [8,9], one obtains

$$\hat{D}^n_x f(x^2) = \hat{D}^n_x\, \mathrm{e}^{x^2\,\hat{\phi}} = H^{(2)}_n(2x\,\hat{\phi}, \hat{\phi})\, \mathrm{e}^{x^2\,\hat{\phi}} = \left\{ n! \sum_{k=0}^{[n/2]} \frac{(2x)^{n-2k}}{(n-2k)!\,k!}\, \hat{\phi}^{n-k} \right\} \mathrm{e}^{x^2\,\hat{\phi}} \tag{1.10}$$

from which, by using Eq. (1.7), we recover Eq. (1.1), see also Appendix A.

The applications of $H^{(2)}_n(x,y)$'s in combinatorics of boson operators can be found in Ref. [10]. We quote for completeness the relation between the $H^{(2)}_n(x,y)$ and the conventional Hermite polynomials $H_n(x)$:

$$H^{(2)}_n(x,y) = (-i)^n y^{n/2} H_n\left(\frac{ix}{2\sqrt{y}}\right). \tag{1.11}$$

In the following we shall bypass the use of Hermite polynomials and shall stick to $H^{(2)}_n(x,y)$. In this way we avoid carrying the imaginary unit inherent from Eq. (1.11). Furthermore, $H^{(2)}_n(x,y)$ permit natural generalizations in the form $H^{(m)}_n(x_1, x_2, \ldots, x_m)$, $m > 2$, see Section 3.

We close this section by stressing that the umbral operator notation as encoded in Eq. (1.2) is entirely equivalent to the use of the derivative. However the umbral notation is very flexible and permits one a very quick demonstration of otherwise involved formulas. We have adopted it in our previous work on the Ramanujan Master Theorem [4] and operational methods [5–7] and continue to use it in the present investigation.

2. An extension of Leibniz formula

In this section we will draw further consequences from the previous formalism. In particular, we will consider the case in which the function $f(x)$ is the product of two functions, obtaining a closed formula which will be recognized as a generalized version of the Leibniz rule.

We start by considering the following function

$$f(x^2) = g(x^2)\,h(x^2) = \left(\mathrm{e}^{x^2\,\hat{\gamma}}\, g_0\right)\left(\mathrm{e}^{x^2\,\hat{\eta}}\, h_0\right), \tag{2.1}$$

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